Ambidexterity in chromatic homotopy theory

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1 Introduction

In [HL13], Hopkins and Lurie proved that certain limits and colimits in $\mathrm{Sp}_{K(n)}$ agree.

Theorem 1.1 ([HL13]). *Let A be an anima that is m-truncated for some integer $m \geqslant -2$ with finitely many connected components and finite homotopy groups. Then for any $F\colon A \to \mathrm{Sp}_{K(n)}$, there is a canonical equivalence*

$$\mathrm{Nm}_A\colon \operatorname*{colim}_A F \xrightarrow{\simeq} \lim_A F \in \mathrm{Sp}_{K(n)} .$$

In this situation, we say that $\mathrm{Sp}_{K(n)}$ is ∞-*semiadditive* (Definition 2.7).

The purpose of this survey is to give a direct proof to the main theorem of [CSY22], which is an analog of Hopkins-Lurie's theorem in $\mathrm{Sp}_{T(n)}$.

Theorem 1.2 ([CSY22, Theorem A]). *The ∞-category $\mathrm{Sp}_{T(n)}$ is ∞-semiadditive.*

In the proof of the theorem, we will also show that $\mathrm{Sp}_{K(n)}$ and $\mathrm{Mod}_{E_n}(\mathrm{Sp}_{K(n)})$ are ∞-semiadditive, which recovers the previous result by Hopkins-Lurie.

The original paper sets up a general machinery to solve the ambidexterity problems. In this survey, we want to depict the core of the proof and prove Theorem 1.2 more directly. Hence, the content of this survey is almost a subset of the original paper, extracting essential details without any diagram chasing and keeping specific to be intuitive.

2 Local systems and ambidexterity

In this section, we set up some fundamental concepts and tools used later in the proof of Theorem 1.2. As noticed in the introduction, in order to make the context more intuitive, we restrict ourselves to the case of ***local systems***, which are ∞-categories of the form $\mathrm{Fun}(A, \mathcal{C})$ for some $A \in \mathrm{An}$ and ∞-category $\mathcal{C}$.

2.1 π-finite anima and the norm map

The proof is based on a previous result by Kuhn.

Theorem 2.1 ([Kuh04]). *The ∞-category $\mathrm{Sp}_{T(n)}$ is 1-semiadditive.*

We want to prove the ∞-semiadditivity of $\mathrm{Sp}_{T(n)}$ by induction. To do this, we first make precise what the index means here. The definition itself is also inductive.

Definition 2.2. Suppose $A \in \mathrm{An}$. We say that A is

1. (-2)***-finite*** if A is contractible.

2. m***-finite*** for some integer $m \geqslant -1$ if $\pi_0(A)$ is finite and all fibers of the diagonal map $\Delta_A \colon A \to A \times A$ are $(m-1)$-finite (for $m \geqslant 0$, this is equivalent to say that A has finitely many connected components, and each of them is m-truncated with finite homotopy groups).

3. π***-finite*** if A is m-finite for some $m \geqslant -2$.

In [HL13], Hopkins and Lurie showed that colimits and limits in $\mathrm{Sp}_{K(n)}$ indexed by a π-finite anima A are canonically isomorphic. In fact, we can generalize this isomorphism to colimits and limits indexed by fibers of a map between anima.

Definition 2.3. Suppose $q \colon A \to B$ is a morphism in An and $m \geqslant -2$ is an integer. We say that q is m***-finite (resp.*** π***-finite)*** if $q^{-1}(b)$ is m-finite (resp. π-finite) for each $b \in B$, where $q^{-1}(b)$ is the (homotopy) fiber of q over b.

Definition 2.4. Suppose $q \colon A \to B$ is a morphism in An and $\mathcal{C}$ is an ∞-category. We say $\mathcal{C}$ ***admits all*** q***-(co)limit*** if $\mathcal{C}$ admits all colimits indexed by $q^{-1}(b)$ for all $b \in B$.

We say $\mathcal{C}$ ***admits all*** m***-finite (co)limit*** if $\mathcal{C}$ admits all q-(co)limit for all m-finite map q.

Similarly, we can define the notion that $F \colon \mathcal{C} \to \mathcal{D}$ preserves all q-(co)limits or m-finite (co)limits.

Note that if $B = \mathrm{pt}$, then q-(co)limit is the same with a (co)limit indexed by A.

Construction 2.5. Given a morphism between anima $q \colon A \to B$ and an ∞-category $\mathcal{C}$, we have the following functors.

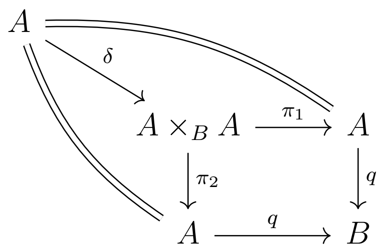
$$\mathrm{Fun}(B, \mathcal{C}) \underset{q_*}{\overset{q_!}{\leftleftarrows}} \xrightarrow{q^*} \mathrm{Fun}(A, \mathcal{C})$$

where q^* is given by composing with q, and $q_!$ and q_* are given by left and right Kan extensions respectively. We will denote the counit and unit map of the adjunction $q^* \dashv q_*$ by c_* and u_* respectively in the rest of the survey. Similar for the adjunction $q_! \dashv q^*$.

Suppose $\delta\colon A \to A \times_B A$ is the diagonal of q and there is an natural isomorphism

$$\mathrm{Nm}_\delta\colon \delta_! \overset{\simeq}{\Rightarrow} \delta_*.$$

We have a wrong way unit map $\mu_\delta\colon \mathrm{Id} \overset{u_\delta}{\to} \delta_*\delta^* \overset{\mathrm{Nm}_\delta^{-1}}{\longrightarrow} \delta_!\delta^*$ and the following commutative diagram.



Composing the base-change map for the square above and μ_δ, we get a wrong way counit map for q by

$$\nu_q\colon q^*q_! \simeq (\pi_2)_!\pi_1^* \overset{\mu_\delta}{\to} (\pi_2)_!\delta_!\delta^*\pi_1^* \simeq \mathrm{Id}\,.$$

Let

$$\mathrm{Nm}_q\colon q_! \to q_*$$

be the mate of ν_q constructed above under the adjunction $q^* \dashv q_*$.

To sum up, we get a norm map on q from the isomorphic norm map on δ. We will use this construction to define ambidexterity and the norm map inductively.

By definition, if q is m-finite, then δ is $(m-1)$-finite.

Definition 2.6. Let $\mathcal{C}$ be an ∞-category, $m \geqslant -2$ be an integer and $q\colon A \to B$ is a map in An. We say q is

1. ***weakly*** m***-***$\mathcal{C}$***-ambidextrous***, if q is m-finite, $\mathcal{C}$ admits all q-(co)limits and either of the two holds:
   - $m = -2$, in which case the inverse of q^* is both a left and right adjoint of q^*. In this case, we define the ***norm map*** $\mathrm{Nm}_q\colon q_! \to q_*$ ***on*** q to be the identity of some inverse of q^*.
   - $m \geqslant -1$ and the diagonal δ of q is $(m-1)$-ambidextrous. In this case, we define the ***norm map*** $\mathrm{Nm}_q\colon q_! \to q_*$ ***on*** q to be the map constructed above from the isomorphism $\mathrm{Nm}_\delta\colon \delta_! \to \delta_*$.

2. m-***$\mathcal{C}$-ambidextrous***, if q is weakly m-$\mathcal{C}$-ambidextrous and the norm map on q is an isomorphism.

   Note that if q is m-$\mathcal{C}$-ambidextrous for some m. Then q is n-$\mathcal{C}$-ambidextrous for all $n \geqslant m$. Therefore, we will omit m and introduce the following definition.

3. ***(weakly) $\mathcal{C}$-ambidextrous***, if q is (weakly) m-$\mathcal{C}$-ambidextrous for some $m \geqslant -2$.

If $B = \mathrm{pt}$, then we will say A is ***(weakly) $\mathcal{C}$-ambidextrous*** if q is and denote Nm_q by Nm_A.

The following is a central notion of this survey.

Definition 2.7. Let $m \geqslant -2$ be an integer. An ∞-category $\mathcal{C}$ is called m-***semiadditive***, if $\mathcal{C}$ admits all m-finite (co)limits and every m-finite map of anima is $\mathcal{C}$-ambidextrous. It is called ∞-***semiadditive*** if $\mathcal{C}$ is m-semiadditive for all integers $m \geqslant -2$.

Example 2.8.

1. Every ∞-category $\mathcal{C}$ is (-2)-semiadditive since the only equivalence class of (-2)-finite anima is a point.

2. Since the only (-1)-finite anima is the empty set, $\mathcal{C}$ is (-1)-semiadditive if and only if the unique map from the initial object to the final object is an isomorphism, if and only if $\mathcal{C}$ is pointed.

3. Since 0-finite anima are finite sets, $\mathcal{C}$ is 0-semiadditive if and only if the canonical maps from finite coproducts to finite products are canonically isomorphic, if and only if $\mathcal{C}$ is semiadditive in the usual sense.

4. Let $A := BG$ for some finite group G be a 1-finite anima. It can be shown that the norm map $\mathrm{Nm}_A \colon X_{hG} \to X^{hG}$ is the classical norm of G. Thus, Nm_A is an isomorphism if and only if the Tate construction vanishes.

2.2 Integration and amenability

Throughout this subsection, let $\mathcal{C}$ be an ∞-category and $q \colon A \to B$ be a $\mathcal{C}$-ambidextrous map.

If B is a point and we have a family of maps $\phi \colon A \to \mathrm{Map}(X, Y)$ for some $X, Y \in \mathcal{C} \simeq \mathrm{Fun}(B, \mathcal{C})$, then we have the following composition of maps

$$X \xrightarrow{\Delta} \lim_A X \xrightarrow{\lim_A \phi} \lim_A Y \xrightarrow{\mathrm{Nm}_A^{-1}} \operatorname*{colim}_A Y \xrightarrow{\nabla} Y.$$

summing up ϕ over A. If A is a finite set, then this composition reduces to the usual addition of maps in a semiadditive category as illustrated in Example 2.8(3). In particular, if $X = Y$ and ϕ is a constant diagram on Id_X, then the sum is the cardinality of A. The definition below is a generalization of this situation to arbitrary B, i.e., a twisted sum of maps via ambidexterity.

Definition 2.9. For every $X, Y \in \mathrm{Fun}(B, \mathcal{C})$, the ***integral map***

$$\int_q : \mathrm{Map}(q^*X, q^*Y) \to \mathrm{Map}(X, Y),$$

is defined as the composition

$$\mathrm{Map}(q^*X, q^*Y) \xrightarrow{q_*} \mathrm{Map}(q_*q^*X, q_*q^*Y) \xrightarrow{\mathrm{Nm}_q^{-1}} \mathrm{Map}(q_*q^*X, q_!q^*Y) \xrightarrow{c_! \circ - \circ u_*} \mathrm{Map}(X, Y).$$

Definition 2.10. For every $X \in \mathrm{Fun}(B, \mathcal{C})$, we define the ***cardinality of*** q ***on*** X to be the map

$$|q|_X := \int_q q^* \mathrm{Id}_X = \int_q \mathrm{Id}_{q^*X} : X \to X.$$

Note that $|q|$ is a natural transformation of $\mathrm{Id}_{\mathcal{C}}$.

Notation. *If $B = \mathrm{pt}$, then we will write $\int_A$ and $|A|$ for $\int_q$ and $|q|$ respectively.*

The integral map enjoys the following important and intuitive properties. The proofs are just diagram-chasing.

Proposition 2.11 (Homogeneity, [CSY22, Proposition 2.1.14])**.** *Let $X, Y, Z \in \mathrm{Fun}(B, \mathcal{C})$.*

1. *For any $f \in \mathrm{Map}(q^*X, q^*Y)$ and $g \in \mathrm{Map}(Y, Z)$, we have*

$$g \circ \left(\int_q f\right) \simeq \int_q (q^*g \circ f).$$

2. *For any $f \in \mathrm{Map}(X, Y)$ and $g \in \mathrm{Map}(q^*Y, q^*Z)$, we have*

$$\left(\int_q g\right) \circ f \simeq \int_q (g \circ q^*f).$$

Proposition 2.12 (Higher Fubini's Theorem, [CSY22, Proposition 2.1.15])**.** *Suppose $q\colon A \to B$ and $p\colon B \to C$ are $\mathcal{C}$-ambidextrous maps. For all $X, Y \in \mathrm{Fun}(C, \mathcal{C})$ and all morphism*

$f\colon q^*p^*X \to q^*p^*Y$, we have

$$\int_p \Big(\int_q f\Big) = \int_{pq} f.$$

Proposition 2.13 ([CSY22, Proposition 2.2.12, Lemma 3.1.2 and Corollary 3.2.7])**.**

1. *Let*

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{s_A} & A \\ \big\downarrow{\scriptstyle \tilde{q}} & & \big\downarrow{\scriptstyle q} \\ \tilde{B} & \xrightarrow{s_B} & B \end{array}$$

   be a pullback square in An*. If q and $\tilde{q}$ are both $\mathcal{C}$-ambidextrous, then for all $X, Y \in \mathrm{Fun}(B, \mathcal{C})$ and $f \in \mathrm{Map}(q^*X, q^*Y)$, we have*

$$s_B^*\left(\int_q f\right) \simeq \int_{\tilde{q}} s_A^* f.$$

   In particular, $s_B^*(|q|_X) \simeq |\tilde{q}|_{s_A^* X}$.

2. *Suppose q is an m-finite map and $\mathcal{C}$ and $\mathcal{D}$ are m-semiadditive ∞-categories. Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a functor that preserves all m-colimits. For all $X, Y \in \mathrm{Fun}(B, \mathcal{C})$ and $f \in \mathrm{Map}(q^*X, q^*Y)$, we have*

$$F\left(\int_q f\right) \simeq \int_q F(f).$$

   In particular, we have $F(|q|_X) \simeq |q|_{F(X)}$.

As mentioned before, we aim to prove the ∞-semiadditivity inductively. Suppose we have proven that $\mathrm{Sp}_{T(n)}$ is m-semiadditive for some $m \geqslant 1$. Pick any $(m+1)$-finite anima A. Note that A is $(m+1)$-finite if and only if ΩA is m-finite. Let $\Omega A \to * \to A$ be the path fibration over A. Heuristically, if we can show that ΩA is kind of "invertible", then we may pass the ambidexterity of $*$ to A. A canonical choice for the definition of the invertibility in the context of ambidexterity should be nothing but the invertibility of the cardinality.

Definition 2.14. We say q is $\mathcal{C}$-***amenable*** if $|q|$ is a natural isomorphism.

If $B = \mathrm{pt}$, then we will say A is ***amenable*** if q is.

The following theorem extends the fact in the representation theory of finite groups that every vector space is a retract of the induced representation via "averaging" over G if $|G|$ is not divisible by the characteristic of the base field.

Theorem 2.15. *If q is amenable, then for any $X \in \mathrm{Fun}(B, \mathcal{C})$, the counit map $c_! \colon q_! q^* X \to X$ admits a section. In particular, every object in $\mathrm{Fun}(B, \mathcal{C})$ is a retract of an object in the essential image of $q_!$.*

Proof. For any $X \in \mathcal{C}$, the composition $q^* X \xrightarrow{u_!} q^* q_! q^* X \xrightarrow{q^* c_!} q^* X$ is homotopic to $\mathrm{Id}_{q^* X}$ by the triangle identity. By definition and Proposition 2.11(1), we have

$$|q|_X = \int_q \mathrm{Id}_{q^* X} \simeq \int_q (q^* c_! \circ u_!) \simeq c_! \circ \int_q u_!.$$

Therefore, $(\int_q u_!) \circ |q|_X^{-1}$ is a section of $c_!$. □

Lemma 2.16. *Let $A \to E \xrightarrow{p} B$ be a fiber sequence of weakly $\mathcal{C}$-ambidextrous anima, where B is connected. If E is $\mathcal{C}$-ambidextrous and A is $\mathcal{C}$-amenable, then B is $\mathcal{C}$-ambidextrous.*

Proof. We first show that p is amenable. Then we use Theorem 2.15 to finish the proof.

Since A is $\mathcal{C}$-ambidextrous, p is $\mathcal{C}$-ambidextrous by [HL13, Proposition 4.3.5(1)]. Let $i \colon \mathrm{pt} \to B$ be a base-point. By Proposition 2.13(1), $i^*(|p|_X) \simeq |A|_{i^* X}$ for all $X \in \mathrm{Fun}(B, \mathcal{C})$. Since B is connected, i^* is conservative. Thus, the amenability of p follows from the amenability of A.

Let $q_B \colon B \to \mathrm{pt}$ be the terminal map. Since E is $\mathcal{C}$-ambidextrous, $q_B p$ is $\mathcal{C}$-ambidextrous by definition. The norm map can be shown to be functorial for composition, i.e.,

$$\mathrm{Nm}_{q_B p} \simeq ((q_B)_* \mathrm{Nm}_p) \circ (\mathrm{Nm}_{q_B} p_!) \colon (q_B)_! p_! \to (q_B)_* p_! \to (q_B)_* p_*.$$

Thus, Nm_{q_B} is an isomorphism on the essential image of $p_!$. By Theorem 2.15, Nm_p is an isomorphism for all $X \in \mathcal{C}$ since isomorphisms are closed under retracts. □

2.3 Monoidal structures

In this subsection, let $(\mathcal{C}, \otimes, \mathbb{1}_{\mathcal{C}})$ be a monoidal ∞-category. Let $q \colon A \to B$ be a weakly $\mathcal{C}$-ambidextrous map such that $\mathcal{C}$ admits all q-colimits and q-limits, and the tensor product distributes over all q-colimits. Thus, $q^* \colon \mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(A, \mathcal{C})$ is monoidal in a canonical way ([Lur17, Example 3.2.4.4]), so $q_!$ is colax monoidal by the dual of [Lur17, Corollary 7.3.2.7].

Proposition 2.17 ([CSY22, Proposition 3.3.1])**.** *For any $X \in \mathrm{Fun}(B, \mathcal{C})$ and $Y \in \mathrm{Fun}(A, \mathcal{C})$,* *the compositions of canonical maps*

$$q_!(Y \otimes (q^*X)) \to (q_!Y) \otimes (q_!q^*X) \xrightarrow{\mathrm{Id}\otimes c_!} (q_!Y) \otimes X$$
$$q_!((q^*X) \otimes Y) \to (q_!q^*X) \otimes (q_!Y) \xrightarrow{c_!\otimes \mathrm{Id}} X \otimes (q_!Y)$$

are both naturally isomorphisms. In particular, $q_!q^*X \simeq X \otimes (q_!q^* \mathbb{1}_{\mathrm{Fun}(B,\mathcal{C})})$ *naturally.*

The above projection formulas suggest that in order to check the ambidexterity at of q in $\mathcal{C}$, it suffices to check it at the tensor unit. To be precise, we have the following proposition.

Proposition 2.18 ([CSY22, Proposition 2.3.4]). *The map* q *is* $\mathcal{C}$*-ambidextrous if and only if* Nm_q *is an isomorphism at* $q^* \mathbb{1}_{\mathrm{Fun}(B,\mathcal{C})}$.

Now let $(\mathcal{D}, \otimes, \mathbb{1}_{\mathcal{D}})$ be another monoidal ∞-category that admits all q-colimits and q-limits, and the tensor product distributes over all q-colimits. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a monoidal functor. Note that

$$q^* \mathbb{1}_{\mathrm{Fun}(B,\mathcal{D})} \simeq q^*F \mathbb{1}_{\mathrm{Fun}(B,\mathcal{C})} \simeq Fq^* \mathbb{1}_{\mathrm{Fun}(B,\mathcal{C})}$$

by the monoidality of F and the commutativity of q^* and F. Hence, we have the following composition of maps

$$\begin{aligned} q_!q^* \mathbb{1}_{\mathrm{Fun}(B,\mathcal{D})} &\xrightarrow{\simeq} q_!Fq^* \mathbb{1}_{\mathrm{Fun}(B,\mathcal{C})} \xrightarrow{\beta_!} Fq_!q^* \mathbb{1}_{\mathrm{Fun}(B,\mathcal{C})} \\ &\xrightarrow{\mathrm{Nm}_q} Fq_*q^* \mathbb{1}_{\mathrm{Fun}(B,\mathcal{C})} \xrightarrow{\beta_*} q_*Fq^* \mathbb{1}_{\mathrm{Fun}(B,\mathcal{C})} \xrightarrow{\simeq} q_*q^* \mathbb{1}_{\mathrm{Fun}(B,\mathcal{D})}, \end{aligned}$$

where $\beta_!$ and β_* are the Beck-Chevalley maps. This map may not be the same with Nm_q on $q^* \mathbb{1}_{\mathrm{Fun}(B,\mathcal{D})}$, but actually it is at most time (this motivates the definition of the *weakly-ambidextrous squares* in [CSY22, Definition 2.2.9]). If F preserves all q-colimits and q-limits, then the Beck-Chevalley maps are isomorphisms by the point-wise formulas of Kan extensions. Therefore, we can transmit ambidexterity of q in $\mathcal{C}$ to the ambidexterity of q in $\mathcal{D}$ through such a functor F.

Lemma 2.19 ([CSY22, Corollary 3.3.2]). *Suppose* $F\colon \mathcal{C} \to \mathcal{D}$ *is an* m*-finite colimit preserving monoidal functor between monoidal categories that admit, and the tensor products distribute over,* m*-finite colimits.*

1. *If* $q\colon A \to B$ *is an* m*-finite,* $\mathcal{C}$*-ambidextrous and weakly* $\mathcal{D}$*-ambidextrous, then* q *is* $\mathcal{D}$*-ambidextrous.*

2. *If $\mathcal{C}$ is m-semiadditive, then $\mathcal{D}$ is m-semiadditive.*

The following definition is a natural notion of (symmetric) monoidal structures for m-semiadditive categories.

Definition 2.20. Suppose $(\mathcal{C}, \otimes, \mathbb{1}_{\mathcal{C}})$ is an m-semiadditive (symmetric) monoidal ∞-category. We say $\mathcal{C}$ is ***m-semiadditively (symmetric) monoidal*** if $\otimes$ distributes over m-finite colimits.

Lemma 2.21. *Suppose $\mathcal{C}$ is m-semiadditively monoidal and A is an m-finite anima.*

1. *For any $X \in \mathcal{C}$, $|A|_X \simeq \mathrm{Id}_X \otimes |A|_{\mathbb{1}_{\mathcal{C}}}$.*

2. *The anima A is $\mathcal{C}$-amenable if and only if $|A|_{\mathbb{1}_{\mathcal{C}}}$ is an isomorphism.*

Proof. It is clear that (2) follows from (1).

For any $X \in \mathcal{C}$, $X \otimes - \colon \mathcal{C} \to \mathcal{C}$ preserves all colimits. Thus, we have

$$|A|_X \simeq |A|_{X \otimes \mathbb{1}_{\mathcal{C}}} \simeq \mathrm{Id}_X \otimes |A|_{\mathbb{1}_{\mathcal{C}}}$$

by Proposition 2.13(2). □

Notation. *For an m-semiadditively symmetric monoidal ∞-category $(\mathcal{C}, \otimes, \mathbb{1}_{\mathcal{C}})$ and an m-finite space A, we will simply denote $|A|_{\mathbb{1}_{\mathcal{C}}}$ by $|A|$. If we want to emphasize the category $\mathcal{C}$, we will write $|A|_{\mathcal{C}}$.*

3 Ambidexterity in chromatic homotopy theory

3.1 Recollection of chromatic homotopy theory

3.1.1 Localizations

In this subsection we recall some notions in chromatic homotopy theory. The results are standard, so we suggest the readers to find the proof in the reference. Let $(\mathrm{Sp}, \otimes, \mathbb{S})$ be the symmetric monoidal ∞-category of spectra.

Recall from [Lur09, Section 5.2.7] that a functor $L : \mathrm{Sp} \to \mathrm{Sp}$ is called a ***localization functor*** if it factors as a composition $\mathrm{Sp} \to \mathrm{Sp}_L \to \mathrm{Sp}$, where the second functor is fully faithful and the first is its left adjoint. We abuse notation and denote by L also the left adjoint $\mathrm{Sp} \to \mathrm{Sp}_L$ itself. We call a map f in Sp an L-equivalence, if $L(f)$ is an isomorphism.

Definition 3.1. A localization functor$L : \mathrm{Sp} \to \mathrm{Sp}$ is called a $\otimes$***-localization*** if L is compatible with the symmetric monoidal structure. That is to say, L-equivalences are closed under tensor product with all objects of C, See [Lur17, Definition 2.2.1.6, Example 2.2.1.7].

Proposition 3.2 ([CSY22, Proposition 5.1.2])**.** *For every* $\otimes$*-localization* $L : \mathrm{Sp} \to \mathrm{Sp}$*, the* ∞*-category* Sp_L *is stable, presentable and admits a structure of a presentably symmetric monoidal* ∞*-category* $(\mathrm{Sp}, \widehat{\otimes}, L\mathbb{S})$ *such that the functor* $L : \mathrm{Sp} \to \mathrm{Sp}_L$ *is symmetric monoidal. Moreover, the inclusion* $\mathrm{Sp}_L \hookrightarrow \mathrm{Sp}$ *admits a canonical lax symmetric monoidal structure. Finally, for all* $X, Y \in \mathrm{Sp}_L$ *we have*

$$X \widehat{\otimes} Y \simeq L(X \otimes Y).$$

For every spectrum $E \in \mathrm{Sp}$, we denote by $L_E : \mathrm{Sp} \to \mathrm{Sp}$ the $\otimes$-localization with essential image the E-local spectra. We denote Sp_{L_E} by Sp_E and $L_E(\mathbb{S})$ by $\mathbb{S}_E$. For a prime p, we shall consider also $\otimes$-localizations $L : \mathrm{Sp}_{(p)} \to \mathrm{Sp}_{(p)}$. The analogous results and notation apply to the p-local case as well.

Proposition 3.3 ([CSY22, Proposition 5.1.3])**.** *Let* $E \in \mathrm{Sp}$ *and let* R *be an* E*-local* $\mathbb{E}_1$*-ring. The* ∞*-category* $\mathrm{Mod}_R^{(E)}$ *of left modules over* R *in the symmetric monoidal* ∞*-category* Sp_E*, is presentable and admits a structure of a presentably symmetric monoidal* ∞*-category. Moreover, we have a free-forgetful adjunction*

$$F_R : \mathrm{Sp}_E \rightleftharpoons \mathrm{Mod}_R^{(E)} : U_R,$$

in which F_R *is symmetric monoidal.*

3.1.2 Morava theories and telescopic localizations

Given an integer $n \geq 0$, let E_n be a 2-periodic ***Morava*** E***-theory*** of height n with coefficients (for $n \geq 1$)

$$\pi_* E_n \simeq \mathbb{Z}_p[[u_1, \ldots, u_{n-1}]][u^{\pm 1}], \quad |u_i| = 0, |u| = 2,$$

and let $K(n)$ be a 2-periodic ***Morava*** K***-theory*** of height n with coefficients (for $n \geq 1$)

$$\pi_* K(n) \simeq \mathbb{F}_p[u^{\pm 1}], \quad |u| = 2.$$

The spectrum E_n admits an $\mathbb{E}_\infty$-ring structure in $\mathrm{Sp}_{K(n)}$ (by [Lur18]). We shall use the notation $\widehat{\mathrm{Mod}}_{E_n}$ for $\mathrm{Mod}_{E_n}^{(K(n))}$.

Definition 3.4. A finite p-local spectrum X, i.e. a compact object in the ∞-category $\mathrm{Sp}_{(p)}$, is said to be of type n, if $K(n) \otimes X \neq 0$ and $K(j) \otimes X = 0$ for $j = 0, 1, \dots, n-1$.

Every type n spectrum $F(n)$ admits a v_n-self map, which is a map

$$v : \Sigma^k F(n) \to F(n),$$

that is an isomorphism on $K(n)_* X$ and zero on $K(j)_* X$ for $j \neq n$. We choose $F(n)$ an $\mathbb{E}_1$-ring spectrum of type n (say, $\underline{\mathrm{Hom}}(F'(n), F'(n))$ for a finite p-local spectrum $F'(n)$ of type n) and let

$$T(n) = v^{-1}F(n) = \varinjlim_k \big(F(n) \xrightarrow{v} \Sigma^{-k}F(n) \xrightarrow{v} \Sigma^{-2k}F(n) \xrightarrow{v} \dots\big)$$

be the telescope on v. The canonical map $F(n) \to T(n)$ exhibits $T(n)$ as the $T(n)$-localization of $F(n)$. Since the functor $L_{T(n)}$ is symmetric monoidal, we can consider $T(n) = L_{T(n)}F(n)$ as an $\mathbb{E}_1$-ring in $\mathrm{Sp}_{T(n)}$. By the Thick Subcategory and Periodicity theorems, the localization $\mathrm{Sp}_{T(n)}$ depends only on the prime p and the height n and in particular is independent of the choice of $F(n)$ and v. It is known that

$$\mathrm{Sp}_{K(n)} \subseteq \mathrm{Sp}_{T(n)} \subseteq \mathrm{Sp}\,.$$

(In fact, due to the latest paper [BHLS23] disproving the telescope conjecture, all inclusions above are strict.)

Thus, both E_n and $K(n)$ are also $T(n)$-local, and so we can consider them as an $\mathbb{E}_\infty$-ring and an $\mathbb{E}_1$-ring in $\mathrm{Sp}_{T(n)}$ respectively.

Morava K-theories are used in the following definition of support:

Definition 3.5. Let $L : \mathrm{Sp}_{(p)} \to \mathrm{Sp}_{(p)}$ be a $\otimes$-localization functor. The (chromatic) ***support*** of L is the set

$$\mathrm{supp}(L) = \{0 \leq n \leq \infty \mid L(K(n)) \neq 0\} \subseteq \mathbb{N} \cup \{\infty\}.$$

For $E \in \mathrm{Sp}_{(p)}$ we denote $\mathrm{supp}(E) = \mathrm{supp}(L_E)$.

Using the above definition, we can state the Nilpotence theorem by Devinatz-Hopkins-Smith. For convenience, we introduce a notion of ”ring” spectrum:

Definition 3.6. A ***weak ring*** is a spectrum $R \in \mathrm{Sp}$, together with a ”unit” map $u : \mathbb{S} \to R$ and a "multiplication" map $R \otimes R \to R$, such that the composition

$$R \xrightarrow{u \otimes \mathrm{Id}} R \otimes R \xrightarrow{\mu} R,$$

is homotopic to the identity. Note that being weak ring is closed under tensor products.

Now we are ready to state the theorem:

Theorem 3.7 (Devinatz-Hopkins-Smith)**.** *Let R be a p-local weak ring. Then $R = 0$ if and only if* $\mathrm{supp}(R) = \varnothing$.

The following argument illustrates how to deduce this form from the standard version.

Proof. Consider the unit map $u : \mathbb{S} \to R$. If $K(n) \otimes R = 0$ for all $0 \neq n \neq \infty$, then by [HS98, Theorem 3(iii)], the map u is smash nilpotent. Namely, $u^{\otimes r} \colon \mathbb{S} \to R^{\otimes r}$ is null for some $r \geq 1$. The commutative diagram

$$\begin{array}{ccc} \mathbb{S} \otimes \mathbb{S} & \xrightarrow{u \otimes u} & R \otimes R \\ {\scriptstyle \mathrm{Id} \otimes u} \downarrow & \overset{u \otimes \mathrm{Id}}{\nearrow} & \downarrow {\scriptstyle \mu} \\ \mathbb{S} \otimes R & \xrightarrow{\mathrm{Id}} & R \end{array}$$

shows that u factors through $u \otimes u$. Applying this iteratively, we can factor u through the null map $u^{\otimes r}$ and deduce that u itself is null. Consequently, the factorization of the identity map of R as the composition

$$R \xrightarrow{u \otimes \mathrm{Id}} R \otimes R \xrightarrow{\mu} R,$$

implies that it is null and thus $R = 0$. □

There are corollaries to the Nilpotence theorem that can be used later for proving the higher semiadditivity of $\mathrm{Sp}_{T(n)}$. We start with a definition.

Definition 3.8. We call a monoidal colimit preserving functor $F : \mathcal{C} \to \mathcal{D}$, between stable presentably monoidal ∞-categories ***nil-conservative***, if for every ring $R \in \mathrm{Alg}(\mathcal{C})$, if $F(R) = 0$ then $R = 0$.

The point of nil-conservativity is that, it gives a criterion of ring homomorphisms that detect invertible elements. We will see it in the next subsection. The Nilpotence Theorem gives us the main example of a nil-conservative functor (See [CSY22, Proposition 5.1.15]):

Proposition 3.9. *Let R be a p-local weak ring. The functor*

$$L : \mathrm{Sp}_R \to \prod_{n \in \mathrm{supp}(R)} \mathrm{Sp}_{K(n)},$$

defined by $K(n)$-localizing at the n-th component, is nil-conservative.

Proof. Suppose $S \in \mathrm{Sp}_R$. If $L(S) = 0$, then $S \otimes K(n) = 0$ for all $n \in \mathrm{supp}(R)$. Meanwhile, $R \otimes K(n) = 0$ for all $n \notin \mathrm{supp}(R)$ by definition. Therefore, $S \otimes R \otimes K(n) = 0$ for all $n \in \mathbb{N} \cup \{\infty\}$. Since weak rings are closed under tensor product, $S \otimes R = 0$ by Theorem 3.7, implying that $S = 0$ in Sp_R. □

Corollary 3.10. *For every $0 \leq n \leq \infty$, the functor*

$$\widehat{E}_n[-] : \mathrm{Sp}_{T(n)} \to \widehat{\mathrm{Mod}}_{E_n}$$

given by the composition

$$\mathrm{Sp}_{T(n)} \xrightarrow{L_{K(n)}} \mathrm{Sp}_{K(n)} \xrightarrow{F_{E_n}} \widehat{\mathrm{Mod}}_{E_n}$$

is nil-conservative, where the first functor is $K(n)$-localization and the second is the free functor.

Proof. By Proposition 3.9 and the fact that $\mathrm{supp}(T(n)) = \{n\}$ ([Rav87, Proposition A.2.13]), $L_{K(n)}$ is nil-conservative. It is also clear that $F_{E_n}(-) \simeq E_n \hat{\otimes} - : \mathrm{Sp}_{K(n)} \to \widehat{\mathrm{Mod}}_{E_n}$ is nil-conservative. Therefore, their composition is nil-conservative. □

3.2 Interlude : reduction of the problem

One of our main goals is proving the ∞-semiadditivity of $\mathcal{C} = \mathrm{Sp}_{T(n)}$. Let us assume by induction that $\mathrm{Sp}_{T(n)}$ is m-semiadditive. To show $\mathrm{Sp}_{T(n)}$ is $(m+1)$-semiadditive, we need to prove that for any $(m+1)$-finite map q of anima, q is $\mathrm{Sp}_{T(n)}$-ambidextrous. By [HL13, Proposition 4.4.16], it suffices to show that

1. $B^{m+1}C_p$ is $\mathrm{Sp}_{T(n)}$-ambidextrous.

To show this, we hope to apply Lemma 2.16, which in turn requires a fiber sequence

$$A \to E \to B$$

of anima, where A and E are both m-finite and A is moreover $\mathrm{Sp}_{T(n)}$-amenable.

We first make an ad hoc definition:

Definition 3.11. We call an anima A ***m*-good** if it is connected, m-finite with $\pi_m(A) \neq 0$, and all homotopy groups of A are finite p-groups.

The reason we call such kind of anima "good" is that they are good candidates for such a fiber sequence indicated above:

Lemma 3.12. *If A be an m-good anima, then A is nilpotent. Consequently, A fits into a fiber sequence*

$$A \to E \to B$$

where E is m-finite and $B = B^{m+1}C_p$.

Proof. By [MP12, Theorem 3.2.2], every connected nilpotent space admits a principal refinement of its Postnikov tower. In the case that A is m-good, it means that there is a principal fibration $B^m H \to A \to E$, where H is a subgroup of $\pi_m(A)$ such that $\pi_1(A)$ acts trivially on H. Since H must be a p-group as a subgroup of $\pi_m(A)$, we may shrink it and assume that $H \cong C_p$. Therefore, we get the desired fiber sequence.

We now show that A is nilpotent. Since every p-group is nilpotent, $\pi_1(A)$ is a nilpotent group. It remains to show that the action of a p-group G on an abelian p-group A is nilpotent.

Note that A admits a filtration $A > pA > p^2A > \cdots > 0$, where each p^nA is a G-group since G acts on A by group homomorphisms. For each $n > 0$, the G-equivariant map $A/pA \to p^nA/p^{n+1}A$ induced by multiplication by p^n is always surjective. Thus, it remains to show that G acts on A/pA nilpotently, i.e., we reduce to the case that A is a $\mathbb{F}_p$-vector space.

Suppose $A \cong \mathbb{F}_p^d$ by choosing a basis for A. Then the G-action gives us a group homomorphism $\rho\colon G \to \mathrm{GL}_d(\mathbb{F}_p)$. Since G is a p-group, $\mathrm{im}(\rho)$ is a p-subgroup in $\mathrm{GL}_d(\mathbb{F}_p)$. It can be shown that

$$|\mathrm{GL}_d(\mathbb{F}_p)| = \prod_{i=0}^{d-1}(p^d - p^i) = p^{\frac{d(d-1)}{2}} \prod_{i=0}^{d-1}(p^{d-i} - 1).$$

Let U be the subgroup of upper-triangular unipotent matrices in $\mathrm{GL}_d(\mathbb{F}_p)$. Then $|U| = p^{\frac{d(d-1)}{2}}$. Thus, U is a Sylow p-subgroup of $\mathrm{GL}_d(\mathbb{F}_p)$. By Sylow's theorem, $\mathrm{im}(\rho)$ is a subgroup of some conjugate of U. By changing a basis for A, we may assume that $\mathrm{im}(\rho) < U$. Suppose $\{e_1, \cdots, e_d\}$ is the basis for A. Then $A > \mathbb{F}_p e_2 \oplus \cdots \oplus \mathbb{F}_p e_d > \cdots > \mathbb{F}_p e_n > 0$ is a filtration of A such that G acts trivially on the associated grading. □

Since $\mathrm{Sp}_{T(n)}$ is m-semiadditive by assumption, according to Lemma 2.21, we are reduced to showing that ([CSY22, Lemma 4.3.6])

2. There exists an m-good anima A, such that $|A|_{\mathbb{1}_{\mathcal{C}}} \in \mathrm{Hom}_{\mathcal{C}}(1, 1)$ is an isomorphism.

For any m-finite anima A and m-semiadditively symmetric monoidal ∞-category $(\mathcal{C}, \otimes, \mathbb{1})$, the cardinality $|A|_{\mathbb{1}}$ is an endomorphism on the unit $\mathbb{1}$. The following definition allow us to regard $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ as the commutative ring $\pi_0\mathbb{S}_{T(n)}$, so that we may apply theories developed on commutative rings:

Definition 3.13. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We denote

$$\mathcal{R}_{\mathcal{C}} = \mathrm{Hom}_{h\mathcal{C}}(\mathbb{1}, \mathbb{1})$$

as a commutative monoid. If $\mathcal{C}$ is 0-semiadditive, then $\mathcal{R}_{\mathcal{C}}$ is naturally a commutative ***rig***(a ring without additive inverse) and if C is stable, then it is a commutative ring. Given a symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$, the induced map $\varphi : \mathcal{R}_{\mathcal{C}} \to \mathcal{R}_{\mathcal{D}}$ is a monoid homomorphism. It is also a rig (resp. ring) homomorphism, when $\mathcal{C}$ and $\mathcal{D}$ are 0-semiadditive (resp. stable) and F is a 0-semiadditive functor.

Thus, we are reduced to showing that

2'. There exists an m-good anima A, such that $|A|_{\mathbb{1}_{\mathcal{C}}} \in \pi_0\mathbb{S}_{T(n)}$ is an invertible element.

Recall the notion of nil-conservativity from Definition 3.8. By [CSY22, Proposition 4.4.4], nil-conservative functors are conservative on the full subcategories of right dualizable objects. In particular, we have

Proposition 3.14. *Let $F : \mathcal{C} \to \mathcal{D}$ be a nil-conservative functor. The induced ring homomorphism $\varphi : \mathcal{R}_{\mathcal{C}} \to \mathcal{R}_{\mathcal{D}}$ detects invertibility. Namely, for any element $r \in \mathcal{R}_{\mathcal{C}} = \mathrm{Hom}_{h\mathcal{C}}(1, 1)$, r is invertible if so is $\varphi(r) \in \mathcal{R}_{\mathcal{D}}$.*

Combining with Corollary 3.10, we are allowed to transport 2' into

3. There exists an m-good anima A, such that $|A| \in \pi_0 E_n$ is invertible.

Remark 3.15. Note that the functor $\widehat{E}_n[-] : \mathrm{Sp}_{T(n)} \to \widehat{\mathrm{Mod}}_{E_n}$ satisfies all conditions in Lemma 2.19. Now, under the assumption that $\mathrm{Sp}_{T(n)}$ is m-semiadditive, we also have the m-semiadditivity of $\widehat{\mathrm{Mod}}_{E_n}$, and thus $|A|$ is well defined.

Recall that E_n has known homotopy group:

$$\pi_0 E_n \simeq \mathbb{Z}_p[[u_1, \ldots, u_{n-1}]],$$

and is way better understood than $\mathbb{S}_{T(n)}$. Moreover, according to [BG16, Lemma 1.33], The image of $h : \pi_0\mathbb{S}_{T(n)} \to \pi_0 E_n$ is contained in $\mathbb{Z}_p \subset \mathbb{Z}_p[[u_1, \ldots, u_{n-1}]] = \pi_0 E_n$. In this case, $|A| \in \pi_0 E_n$ is invertible if and only if the p-adic valuation $v_p(|A|)$ is zero.

Now we are reduced to the problem

4. There exists an m-good anima A, such that $v_p(|A|)$ is zero.

4 Stable additive derivations

4.1 Equivariant power

Let $\mathcal{C}$ be a symmetric monoidal ∞-category and p a prime. In this subsection, we construct a functor $\Theta^p : \mathcal{C} \to \mathrm{Fun}(BC_p, \mathcal{C})$, which is additive in the sense that it admits a formula of the form

$$\Theta^p(f+g) = \Theta^p(f) + \Theta^p(g) + \text{"induced terms"},$$

under the assumption that $\mathcal{C}$ is 0-semiadditive.

Definition 4.1. Let $X \in \mathcal{C}$ be an object in a symmetric monoidal ∞-category, the tensor power $X^{\otimes p}$ admits a natural action of the cyclic group $C_p \subseteq \Sigma_p$ (by permuting factors). When $\mathcal{C} = \mathrm{An}$ is the cartesian symmetric monoidal ∞-category of anima, we denote the above homotopy orbit by $X \wr C_p = (X^p)_{hC_p}$.

Lemma 4.2. *The functor* $(-) \wr C_p : \mathrm{An} \to \mathrm{An}$ *preserves fiber products.*

Proof. See [CSY22, Lemma 3.4.1]. □

The construction $(-) \wr C_p$ induces a functor

$$(-)^p_{hC_p} : \mathrm{Fun}(A, \mathcal{C}) \to \mathrm{Fun}((A^p)_{hC_p}, (\mathcal{C}^p)_{hC_p}).$$

Using this have the following:

Definition 4.3. Given a symmetric monoidal ∞-category $\mathcal{C}$, we define the functor

$$\Theta^p_A : \mathrm{Fun}(A, \mathcal{C}) \to \mathrm{Fun}(A \wr C_p, \mathcal{C})$$

to be the composition $(-)^p_{hC_p}$ with

$$(\mathcal{C}^p)_{hC_p} \to (\mathcal{C}^p)_{h\Sigma_p} \xrightarrow{\otimes} \mathcal{C}.$$

When the context is clear, we will suppress the subscript A.

The first core feature of this functor is that it commutes with integrals ([CSY22, Theorem 3.4.8]):

Theorem 4.4. *Let $\mathcal{C}$ be an m-semiadditively symmetric monoidal ∞-category and $q : A \to B$ an m-finite map of spaces. For every $X, Y \in \mathrm{Fun}(B, \mathcal{C})$ and $f : q^*X \to q^*Y$, we have*

$$\Theta^p_B\Big(\int_q f\Big) = \int_{q \wr C_p} \Theta^p_A(f) \in \mathrm{Hom}_{h\,\mathrm{Fun}(B\wr C_p,\mathcal{C})}(\Theta^p(X), \Theta^p(Y)).$$

Remark 4.5. The above equation requires a specification of the "ambidextrous" square given by the naturality of Θ^p:

$$\begin{array}{ccc} \mathrm{Fun}(A, \mathcal{C}) & \xrightarrow{\Theta^p_A} & \mathrm{Fun}(A \wr C_p, \mathcal{C}) \\ \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle (q\wr C_p)^*} \\ \mathrm{Fun}(B, \mathcal{C}) & \xrightarrow{\Theta^p_B} & \mathrm{Fun}(B \wr C_p, \mathcal{C}) \end{array}.$$

(See [CSY22, Definition 2.2.9, Proposition 2.2.12] for the definition and properties of ambidextrous squares.)

To study the additivity of Θ, we confine ourselves to the case

$$q : \mathrm{pt} \sqcup \mathrm{pt} \to \mathrm{pt}\,.$$

Let $f, g : X \to Y$ be two maps in $\mathcal{C}$, such that the pair (f, g) can be identified with a map $q^*X \to q^*Y$ in $\mathrm{Fun}(\mathrm{pt} \sqcup \mathrm{pt}, \mathcal{C})$ and thus the left-hand side of the equation appeared in Theorem 4.4 can be identified with $\Theta^p_{\mathrm{pt}}(f+g)$. So our goal for now is to describe the commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}(\mathrm{pt},\mathcal{C}) & \xrightarrow{\Theta^p_{\mathrm{pt}}} & \mathrm{Fun}(\mathrm{pt}\wr C_p,\mathcal{C}) \\ \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle (q\wr C_p)^*} \\ \mathrm{Fun}(\mathrm{pt}\sqcup\mathrm{pt},\mathcal{C}) & \xrightarrow{\Theta^p_{\mathrm{pt}\sqcup\mathrm{pt}}} & \mathrm{Fun}((\mathrm{pt}\sqcup\mathrm{pt})\wr C_p,\mathcal{C}) \end{array}$$

explicitly and thus to unwind the right hand-side

$$\int_{q\wr C_p}\Theta^p_{\mathrm{pt}\sqcup\mathrm{pt}}(f,g)\in \mathrm{Hom}_{\mathrm{Fun}((\mathrm{pt}\sqcup\mathrm{pt})\wr C_p,\mathcal{C})}(\Theta^p(X),\Theta^p(Y)).$$

Let S be the set

$$S=\{w\in\{x,y\}^p | w\neq x^p,y^p\},$$

with x,y formal variables and let $\bar{S}$ be the set of orbits of S under the action of C_p by cyclic shift. We have an equivalence of anima

$$(\mathrm{pt}\sqcup\mathrm{pt})\wr C_p\simeq BC_p\sqcup BC_p\sqcup\bar{S},$$

and therefore an equivalence of ∞-categories

$$\mathrm{Fun}((\mathrm{pt}\sqcup\mathrm{pt})\wr C_p,\mathcal{C})\simeq \mathcal{C}^{BC_p}\times\mathcal{C}^{BC_p}\times\prod_{\bar{w}\in\bar{S}}\mathcal{C}.$$

Choosing a base point map $e:\mathrm{pt}\to BC_p$, we see that up to homotopy, we have

$$q\wr C_p=(\mathrm{Id},\mathrm{Id},e,\dots,e).$$

Similiarly, under the above identification, the functor $\Theta^p_{\mathrm{pt}\sqcup\mathrm{pt}}$ can also be identified with a functor

$$\Phi:\mathcal{C}\times\mathcal{C}\to\mathcal{C}^{BC_p}\times\mathcal{C}^{BC_p}\times\prod_{\bar{w}\in\bar{S}}\mathcal{C},$$

which can be described as follows:

For each element $w=(w_1,w_2,\dots,w_p)\in\{x,y\}^p$, we define a functor $w(-,-):\mathcal{C}\times\mathcal{C}\to\mathcal{C}$, which, informally speaking, is given by

$$w(X,Y) = Z_1 \otimes Z_2 \otimes \cdots \otimes Z_p, \qquad Z_i = \begin{cases} X & \text{if } w_i = x \\ Y & \text{if } w_i = y \end{cases}.$$

Then we have

Lemma 4.6 ([CSY22, Lemma 3.4.9]). *The functor Φ is naturally equivalent to*

$$(\Theta^p \circ p_1, \Theta^p \circ p_2, \{w(-,-)\}_{\bar{w}\in\bar{S}}),$$

where $p_i : \mathcal{C} \times \mathcal{C}$ denotes the projection to the i-th component(it does not matter which representative w we take for $\bar{w} \in \bar{S}$).

Using this, we may compute

$$\begin{aligned} \Theta^p(f+g) = \Theta^p(\int_q (f,g)) &= \int_{q\wr C_p} \Theta^p_{\mathrm{pt}\sqcup\mathrm{pt}}(f,g) \\ &= \int_{(\mathrm{Id},\mathrm{Id},e,\ldots,e)} \big(\Theta^p(f), \Theta^p(g), \{w(f,g)\}_{\bar{w}\in\bar{S}}\big) \\ &= \Theta^p(f) + \Theta^p(g) + \sum_{\bar{w}\in\bar{S}} \big(\int_e \{w(f,g)\}\big). \end{aligned}$$

This gives the desired formula([CSY22, Proposition 3.4.10]).

4.2 Additive p-derivation in number theory

The functor Θ^p described in the last subsection can be used to define an operation α which helps us to reduce the p-adic valuation of the cardinality for an m-good anima. Finally, we will find an m-good anima A whose cardinality $|A|$ lies in $\mathbb{Z}_p^\times \subset \mathbb{Z}_p[[u_1,\ldots,u_{n-1}]] \simeq \pi_0(E_n)$.

Let's start with the Fermat quotient in number theory:

Example 4.7. Let R be a subring of $\mathbb{Q}$ (or $\mathbb{Q}_p$). Then the expression

$$\bar{\delta}(x) = \frac{x - x^p}{p}$$

satisfies

1. $\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p+y^p-(x+y)^p}{p}$ for all $x, y \in R$.

2. $\delta(0) = \delta(1) = 0$.

3. $v_p(\bar{\delta}(x)) = v_p(\frac{x-x^p}{p}) = v_p(x) - 1$ for every such x that $0 < v_p(x) < \infty$, where $v_p : R \to \mathbb{Z} \cup \{\infty\}$ is the p-adic valuation.

We treat the Fermat quotient as a prototype of an additive p-derivation, that is to say, an operation satisfying condition 1 and 2 as in Example 4.7.

Definition 4.8. Let R be a commutative ring. An additive p-derivation on R is a function of sets

$$\delta : R \to R,$$

that satisfies :

1. (additivity) $\delta(x + y) = \delta(x) + \delta(y) + \frac{x^p+y^p-(x+y)^p}{p}$ for all $x, y \in R$.
2. (normalization) $\delta(0) = \delta(1) = 0$.

The pair (R, δ) is called a semi-δ-ring. A semi-δ-ring homomorphism from (R, δ) to (R', δ') is a homomorphism $f : R \to R'$, that satisfies $f \circ \delta = \delta' \circ f$.

Remark 4.9. 1. The expression $\frac{x^p+y^p-(x+y)^p}{p}$ is well defined even when R is p-torsion, as it is actually a polynomial with integer coefficients.

Definition 4.10. Let R be a commutative ring. Let $\phi_0 : \mathbb{Z} \to R$ be the unique ring homomorphism and let S_R be the set of primes p, such that $\phi_0(p) \in R^\times$. We denote

$$\mathbb{Q}_R = \mathbb{Z}[S_R^{-1}] \subseteq \mathbb{Q}$$

and $\phi : \mathbb{Q}_R \to R$ the unique extension of ϕ_0. We call an element $x \in R$ rational if it is in the image of ϕ. By Example 4.7, $(\mathbb{Q}_R, \bar{\delta})$ is a semi-δ-ring.

Though we do not require the derivation δ in a semi-δ-ring to be multiplicative, the following lemma plays a similar role to the ”multiplication formula” for the Fermat quotient. See [CSY22, Lemma 4.1.9].

Lemma 4.11. *Let (R, δ) be a semi-δ-ring and let $\bar{\delta}$ denote the Fermat quotient on $\mathbb{Q}_R$. For all $t \in \mathbb{Q}_R$ and $x \in R$, we have*

$$\delta(tx) = t\delta(x) + \bar{\delta}(t)x^p.$$

When R is p-local, Lemma 4.11 has the following pleasant consequence. The proofs are left to readers as exercises, or see [CSY22, Proposition 4.1.10, Proposition 4.1.11].

Proposition 4.12. *Let (R, δ) be a p-local semi-δ-ring. If $x \in R$ is torsion, then x is nilpotent.*

Hint. Compare to the explanation below Example 4.7.

Proposition 4.13. *Let (R, δ) be a non-zero p-local semi-δ-ring. The map $\phi : \mathbb{Q}_R \to R$ is an injective semi-δ-ring homomorphism. In particular, $\bar{\delta}$ is the unique additive p-derivation on $\mathbb{Q}_R$.*

Hint. Apply Lemma 4.11 to $x = 1$.

4.3 The Alpha operation

In this subsection, we introduce an additive operation $\alpha : R_\mathcal{C} \to R_\mathcal{C}$ for a stable m-semiadditively symmetric monoidal ∞-category $\mathcal{C}$ such that $|A \wr C_p| = \alpha(|A|)$, $1 \leq m \leq \infty$. In particular, $\alpha(0) = 0$ and $\alpha(1) = |BC_p|$, so $\delta : R_\mathcal{C} \to R_\mathcal{C}, x \mapsto |BC_p|x - \alpha(x)$ is an additive p-derivation, making $(R_\mathcal{C}, \delta)$ a semi-δ-ring.

Throughout the section we denote

$$\mathrm{pt} \xrightarrow{e} BC_p \xrightarrow{r} \mathrm{pt}\,.$$

We first drop out the assumption $\mathcal{C}$ being stable, so that $R = \mathrm{Hom}_{h\mathcal{C}}(X, Y)$ only forms a commutative rig for $X \in \mathrm{coCAlg}(\mathcal{C})$ and $Y \in \mathrm{CAlg}(\mathcal{C})$. The commutative coalgebra and commutative algebra structures, on X and Y respectively, provide symmetric comultiplication and multiplication maps:

$$\bar{t}_X : X \to (X^{\otimes p})^{hC_p} = r_*\Theta^p(X)$$

$$\bar{m}_Y : r_!\Theta^p(Y) = (Y^p)_{hC_p} \to Y.$$

Definition 4.14. Let $\mathcal{C}$ be a 1-semiadditively symmetric monoidal ∞-category and let

$$X \in \mathrm{coCAlg}(\mathcal{C}), Y \in \mathrm{CAlg}(\mathcal{C}).$$

1. Given $g : \Theta^p(X) \to \Theta^p(Y)$, we define $\bar{\alpha}(g) : X \to Y$ to be either of the compositions

in the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\bar{t}_X} & r_*\Theta^p(X) & \xrightarrow{\mathrm{Nm}_r^{-1}} & r_!\Theta^p(X) & & \\ & & {\scriptstyle g}\downarrow & & {\scriptstyle g}\downarrow & & \\ & & r_*\Theta^p(Y) & \xrightarrow{\mathrm{Nm}_r^{-1}} & r_!\Theta^p(Y) & \xrightarrow{\bar{m}_Y} & Y \end{array}$$

2. Given $f : X \to Y$, we define $\alpha(f) = \bar{\alpha}(\Theta^p(f))$.

Lemma 4.15. *Let $\bar{\alpha} : \pi_0 \operatorname{Map}_{\operatorname{Fun}(BC_p,\mathcal{C})}(\Theta^p X, \Theta^p Y) \to \pi_0 \operatorname{Map}_{\mathcal{C}}(X, Y)$ and $\alpha : \pi_0 \operatorname{Map}_{\mathcal{C}}(X, Y) \to \pi_0 \operatorname{Map}_{\mathcal{C}}(X, Y)$ as in Definition 4.14.*

1. *The map $\bar{\alpha}$ is additive, since so is r_*.*

2. *Given maps $Y \to Y'$ and $X' \to X$ of commutative algebras and coalgebras respectively, for every map $f : X \to Y$, we have*

$$\alpha(g \circ f \circ h) = g \circ \alpha(f) \circ h \in \operatorname{Hom}_{h\mathcal{C}}(X', Y').$$

3. *Let $\mathcal{D}$ be a 1-semiadditively symmetric monoidal ∞-category and $F : \mathcal{C} \to \mathcal{D}$ a 1-semiadditive symmetric monoidal functor. The induced map of commutative rigs*

$$\operatorname{Hom}_{h\mathcal{C}}(X, Y) \to \operatorname{Hom}_{h\mathcal{D}}(FX, FY)$$

commutes with the operation α.

Our next goal is to discuss the "addition formula" on α. We first introduce a technical lemma:

Lemma 4.16. *Let $\mathcal{C}$ be a 1-semiadditively symmetric monoidal ∞-category and let $X \in \operatorname{coCAlg}(\mathcal{C}), Y \in \operatorname{CAlg}(\mathcal{C})$. For every map*

$$h : X^{\otimes p} = e^*\Theta^p(X) \to e^*\Theta^p(Y) = Y^{\otimes p},$$

the map $\bar{\alpha}(\int_e h)$ is homotopic to the composition

$$X \xrightarrow{e^* t_X} X^{\otimes p} \xrightarrow{h} Y^{\otimes p} \xrightarrow{e^* m_Y} Y.$$

Using this lemma, we are able to deduce the desired "addition formula":

Theorem 4.17. *Let $\mathcal{C}$ be a 1-semiadditive symmetric monoidal ∞-category and let*

$$X \in \mathrm{coCAlg}(\mathcal{C}), Y \in \mathrm{CAlg}(\mathcal{C}).$$

For every $f, g : X \to Y$, we have

$$\alpha(f+g) = \alpha(f) + \alpha(g) + \frac{(f+g)^p - f^p - g^p}{p} \in \mathrm{Hom}_{h\mathcal{C}}(X, Y).$$

Proof. First we recall that $\bar{\alpha}$ is additive, so we have

$$\begin{aligned}\alpha(f+g) &= \bar{\alpha}(\Theta^p(f+g)) \\ &= \bar{\alpha}(\Theta^p(f) + \Theta^p(g) + \sum_{\bar{w}\in\bar{S}} \big(\int_e \{w(f,g)\}\big) \\ &= \bar{\alpha}(\Theta^p(f)) + \bar{\alpha}(\Theta^p(g)) + \sum_{\bar{w}\in\bar{S}} \bar{\alpha}(\int_e \{w(f,g)\}).\end{aligned}$$

By Lemma 4.16, the map $\bar{\alpha}(\int_e \{w(f,g)\})$ is homotopic to the composition

$$X \xrightarrow{e^* t_X} X^{\otimes p} \xrightarrow{w(f,g)} Y^{\otimes p} \xrightarrow{e^* m_Y} Y.$$

This is by definition $f^{w_x} g^{w_y}$, where w_x and w_y are the number of x-s and y-s in w respectively and this completes the proof. □

Then we apply the above theory to the case where $X = Y = \mathbf{1}$ is the unit of a symmetric monoidal ∞-category $\mathcal{C}$. Recall that $\mathbf{1}$ is canonically both a commutative algebra and a commutative coalgebra, and we denote $\mathcal{R}_{\mathcal{C}} = \mathrm{Hom}_{h\mathcal{C}}(\mathbf{1}, \mathbf{1})$.

Note that the symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{C})$ is cocartesian, as tensor products there coincide with coproducts. Thus, any group action on $\mathbf{1}$ as an initial object in $\mathrm{CAlg}(\mathcal{C})$ must be trivial. Also, the forgetful functor $U : \mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ is symmetric monoidal, so the Σ_p-action on $\mathbf{1}$ in $\mathcal{C}$ is induced by that in $\mathrm{CAlg}(\mathcal{C})$, which must also be trivial. In other words, $\Theta^p(\mathbf{1}) = r^*\mathbf{1}$.

Now, look at the "symmetric" multiplication and comultiplication maps

$$\bar{t}_{\mathbf{1}} : \mathbf{1} \to r_* \Theta^p(\mathbf{1}),\ \bar{m}_{\mathbf{1}} : r_! \Theta^p(\mathbf{1}) \to \mathbf{1}.$$

It is natural (why not?) to guess these maps are (equivalent to) the unit map and the counit map

$$u_* : \mathbf{1} \to r_* r^* \mathbf{1},\ c_! : r_! r^* \mathbf{1} \to \mathbf{1}$$

respectively. Indeed, it is the case, see [CSY22, Lemma 4.2.9, Lemma 4.2.10].

Consequently, we may describe the effect of α on any element of $\mathcal{R}_{\mathcal{C}}$ using the integral operation:

Proposition 4.18. *Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a 1-semiadditively symmetric monoidal ∞-category. For every $f \in \mathcal{R}_{\mathcal{C}}$, we have*

$$\alpha(f) = \int_{BC_p} \Theta^p(f) \in \mathcal{R}_{\mathcal{C}}.$$

In particular, we get an explicit formula for those maps of the form $\alpha(|A|)$:

Theorem 4.19 ([CSY22, 4.2.12])**.** *Let $\mathcal{C}$ be an m-semiadditively symmetric monoidal ∞-category for $m \geq 1$. For every m-finite anima A, we have*

$$\alpha(|A|) = |A \wr C_p|.$$

In particular, $|BC_p| = \alpha(|\operatorname{pt}|) = 1 \in \mathcal{R}_{\mathcal{C}}$.

Proof. Note that this is an application of the Higher Fubini's Theorem(Proposition 2.12). Namely, consider the maps

$$q : A \to \operatorname{pt},\ r : BC_p \to \operatorname{pt},$$

we have

$$\alpha(|A|) = \bar{\alpha}(\Theta^p(|A|)) = \int_r \Theta^p(\int_q \operatorname{Id}_{\mathbf{1}}) = \int_r \int_{q \wr C_p} \operatorname{Id}_{\mathbf{1}} = \int_{r(q \wr C_p)} \operatorname{Id}_{\mathbf{1}} = |A \wr C_p|.$$

The last claim follows immediately. □

Finally, we define an additive p-derivation on the commutative ring $\mathcal{R}_{\mathcal{C}}$ for any stable 1-semiadditively symmetric monoidal ∞-category $\mathcal{C}$.

Definition 4.20. Let $\mathcal{C}$ be a stable 1-semiadditively symmetric monoidal ∞-category. We define an operation $\delta : \mathcal{R}_{\mathcal{C}} \to \mathcal{R}_{\mathcal{C}}$ by

$$\delta(f) = |BC_p|(f) - \alpha(f).$$

Proposition 4.21. *The operation δ in Definition 4.20 is an additive p-derivation on $\mathcal{R}_{\mathcal{C}}$.*

Proof. The additivity follows from Theorem 4.17 and the normalization follows from Theorem 4.19. □

4.4 Application to $\widehat{\mathrm{Mod}}_{E_n}$

In this subsection, we are back to our original goal for higher semi-additivity of $\mathrm{Sp}_{T(n)}$. Recall that we have reduced the problem to finding an m-finite anima A whose cardinality $|A| \in \mathbb{Z}_p \subset \pi_0 E_n$ is invertible, or equivalently, has p-adic valuation 0.

Proposition 4.22. *Let $1 \leq m < \infty$ and let $h : \pi_0 \mathbb{S}_{T(n)} \to \pi_0 E_n$ be the ring homomorphism induced by $\widehat{E}_n[-] : \mathrm{Sp}_{T(n)} \to \widehat{\mathrm{Mod}}_{E_n}$ as in Corollary 3.10. If $\mathrm{Sp}_{T(n)}$ is m-semiadditive (which implies that so is E_n), then there exists an m-good anima A, such that $|A|$ is invertible as an element in $\mathbb{Z}_p \subset \mathbb{Z}_p[[v_1, \ldots, v_{n-1}]] = \pi_0 E_n$.*

Proof. We start with $B^m C_p$. By the computation in appendix A, we have $|B^m C_p| = p^{\binom{n-1}{m}}$ whenever it is well-defined. In particular, it is rational and has p-adic valuation $0 \leq v_p(|B^m C_p|) < \infty$. It therefore suffices to show that given an m-good anima A with $0 < v(A) = v_p(|A|) < \infty$, there exists another m-good anima A' such that $v(A') = v(A) - 1$. We compute

$$\delta(|A|) = |BC_p||A| - \alpha(|A|) = |BC_p||A| - |A \wr C_p|,$$

then

$$|A \wr C_p| = |BC_p||A| - \delta(|A|).$$

Note that the only non-invertible prime number in $\pi_0 E_n$ is p, so $v(A) \leq v_p(|BC_p||A|)$. Moreover, $v_p(\delta(|A|)) = v(A) - 1$ by the fact that $|A| \in \mathbb{Z}_p$, where by Proposition 4.13 the restriction of δ coincides with the Fermat quotient as stated in Example 4.7. It follows that $v(A \wr C_p) = v(A) - 1$. Clearly, $A \wr C_p$ is m-good, and this completes the proof. □

The above proof justifies that if $\mathrm{Sp}_{T(n)}$ is m-semiadditive, then $\mathrm{Sp}_{T(n)}$ is $(m+1)$-semiadditive. Finally, we have

Theorem 4.23 ([CSY22, Theorem 5.3.1]). *For all $n \geq 0$, the ∞-categories $\mathrm{Sp}_{T(n)}$ and $\widehat{\mathrm{Mod}}_{E_n}$ are ∞-semiadditive.*

Applying Lemma 2.19 to $L_{K(n)}\colon \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{K(n)}$, we recover the result of [HL13].

Corollary 4.24. *For all $0 \leqslant n < \infty$, $\mathrm{Sp}_{K(n)}$ is ∞-semiadditive.*

Suppose R is a weak ring. It is natural to ask about the semiadditivity of Sp_R. We will finally state (without proof) another result in [CSY22], giving a complete answer to this question and showing the relationship between Sp_R, $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{T(n)}$.

Theorem 4.25 ([CSY22, cf. Theorem 5.4.7]). *Let R be a nonzero p-local weak ring. The following are equivalent:*

1. *There exists a (necessarily unique) integer $n \geqslant 0$, such that $\mathrm{Sp}_{K(n)} \subset \mathrm{Sp}_R \subset \mathrm{Sp}_{T(n)}$.*
2. *Either $\mathrm{Sp}_R = \mathrm{Sp}_{H\mathbb{Q}}$, or $\Omega^\infty\colon \mathrm{Sp}_R \to \mathrm{An}_*$ admits a retract.*
3. *Sp_R is ∞-semiadditive.*
4. *Sp_R is 1-semiadditive.*
5. *$\mathrm{supp}(R) = \{n\}$ for some $0 \leqslant n < \infty$.*

A Computing $|B^m C_p|_{\widehat{\mathrm{Mod}}_{E_n}}$

In this appendix, we compute $|B^m C_p|_{\widehat{\mathrm{Mod}}_{E_n}}$ indirectly via the dimension of $B^m C_p$ in $\widehat{\mathrm{Mod}}_{E_n}$, which can be deduced from a result of Ravenel-Wilson on $K(n)_*(B^m C_p)$.

Let us define the notion of dimension of dualizable objects and compute $\dim_{\widehat{\mathrm{Mod}}_{E_n}}(B^m C_p)$ firstly.

Definition A.1. Suppose $(\mathcal{C}, \otimes, \mathbb{1}_{\mathcal{C}})$ is a symmetric monoidal ∞-category and $X \in \mathcal{C}$ is dualizable. The ***dimension of*** X ***in*** $\mathcal{C}$ $\dim_{\mathcal{C}}(X) \in \mathrm{End}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}})$ is the following composition

$$\mathbb{1}_{\mathcal{C}} \xrightarrow{\mathrm{coev}} X \otimes X^\vee \xrightarrow{\mathrm{swap}} X^\vee \otimes X \xrightarrow{\mathrm{ev}} \mathbb{1}_{\mathcal{C}}\,.$$

Let $q\colon A \to \mathrm{pt}$ be the terminal map in An. We say that A is ***dualizable in*** $\mathcal{C}$ if $q_! q^* \mathbb{1}_{\mathcal{C}}$ is dualizable in $\mathcal{C}$ and we denote

$$\dim_{\mathcal{C}}(A) := \dim_{\mathcal{C}}(q_! q^* \mathbb{1}_{\mathcal{C}}).$$

Lemma A.2. *Let* $n \geqslant 0$ *and* $X \in \mathrm{An}$. *If*

$$\dim_{\mathbb{F}_p}(K(n)_0(X)) = d < \infty \quad \text{and} \quad K(n)_1(X) = 0,$$

then X *is dualizable in* $\widehat{\mathrm{Mod}}_{E_n}$ *and*

$$\dim_{\widehat{\mathrm{Mod}}_{E_n}}(X) = d.$$

Proof. Let $q\colon X \to \mathrm{pt}$ be the terminal map. The lemma follows from [HL13, Proposition 3.4.3(1)], which says that there is an isomorphism in $\widehat{\mathrm{Mod}}_{E_n}$

$$q_! q^* \mathbb{1}_{\widehat{\mathrm{Mod}}_{E_n}} \simeq q_! q^* E_n \simeq L_{K(n)}(E_n \otimes \Sigma^\infty_+ X) \simeq E_n^d. \qquad \square$$

Corollary A.3. *For all* $m \in \mathbb{Z}_{\geqslant 0}$, *we have*

$$\dim_{\widehat{\mathrm{Mod}}_{E_n}}(B^m C_p) = p^{\binom{n}{m}}.$$

Proof. By [RW80, Theorem 9.2], we have

$$\dim_{\mathbb{F}_p} K(n)_0(B^m C_p) = p^{\binom{n}{m}} \quad \text{and} \quad K(n)_1(B^m C_p) = 0.$$

Hence, the result follows from the above lemma. $\square$

Next, we have to deduce the relationship between the dimension and the cardinality of m-finite anima. Note that both of them are preserved by colimit-preserving symmetric monoidal functors. It turns out that the situation in m-semiadditively symmetric monoidal ∞-categories can be reduced to a universal and familiar category.

Let $\mathcal{S}_m^m$ be *the* ∞*-category of spans of* m*-finite anima*. Roughly speaking,

- The objects of $\mathcal{S}_m^m$ are m-finite anima.

- A morphism from A to B is a span $A \leftarrow E \rightarrow B$, where E is also m-finite.
- Composition, up to homotopy, is given by pullback of spans.

By [Har20, Section 2.2], $\mathcal{S}_m^m$ can be promoted to a symmetric monoidal ∞-category by taking the Cartesian product of anima levelwisely.

Theorem A.4 (Harpaz, [Har20, Corollary 5.8]). *We have that the ∞-category $\mathcal{S}_m^m$ is initial among m-semiadditively symmetric monoidal ∞-category, i.e., for any m-semiadditively symmetric monoidal ∞-category $\mathcal{C}$, up to homotopy, there exists a unique m-colimit preserving symmetric monoidal functor $F_{\mathcal{C}} \colon \mathcal{S}_m^m \to \mathcal{C}$, whose underlying functor is given by $A \mapsto q_! q^* \mathbb{1}_{\mathcal{C}}$, where $q \colon A \to \mathrm{pt}$ is the terminal map.*

Thus, by Proposition 2.13(2),

$$\dim_{\mathcal{C}}(A) = F_{\mathcal{C}}(\dim_{\mathcal{S}_m^m}(A)) \quad \text{and} \quad |B|_{\mathbb{1}_{\mathcal{C}}} = F_{\mathcal{C}}(|B|_{\mathrm{pt}})$$

for any m-finite anima A and B.

The relationship is easy in $\mathcal{S}_m^m$.

Proposition A.5. *Every m-finite anima A is self-dual in $\mathcal{S}_m^m$ and satisfies*

$$\dim_{\mathcal{S}_m^m}(A) \simeq (\mathrm{pt} \leftarrow LA \rightarrow \mathrm{pt}) \simeq |LA|,$$

where LA is the free loop space.

Proof. It is straight forward to check that

$$\mathrm{ev} \colon A \times A \xleftarrow{\Delta} A \rightarrow \mathrm{pt} \quad \text{and} \quad \mathrm{coev} \colon \mathrm{pt} \leftarrow A \xrightarrow{\Delta} A \times A$$

satisfy the triangle identity, so they exhibit A as a self-dual object in $\mathcal{S}_m^m$. Moreover, note that $\mathrm{ev} \circ \mathrm{swap} \simeq \mathrm{ev}$. Thus, $\dim(A) \simeq \mathrm{ev} \circ \mathrm{coev}$. The first equivalence comes from the following pullback square in An and the composition rule in $\mathcal{S}_m^m$.

$$\begin{array}{ccc} LA & \longrightarrow & A \\ \big\downarrow & \lrcorner & \big\downarrow \\ A & \longrightarrow & A \times A \end{array}$$

For the second equivalence, for any m-finite anima B, $|B|$ in $\mathcal{S}_m^m$ is computed by By [Har20, Proposition 2.9], m-finite colimits in $\mathcal{S}_m^m$ can be computed as in An, so we have $\mathrm{colim}_B\,\mathrm{pt} \simeq B$. Thus, $|B| \simeq (\mathrm{pt} \leftarrow B \rightarrow \mathrm{pt})$. Taking $B = LA$ finishes the proof. □

Lemma A.6. *If $A \simeq \mathrm{Map}_{\mathrm{An}_*}(B, C)$ for some connected $B \in \mathrm{An}_*$ and arbitrary $C \in \mathrm{An}_*$, then $LA \simeq A \times \Omega A$.*

Proof. We have

$$\begin{aligned} LA &\simeq \mathrm{Map}_{\mathrm{An}}(S^1, \mathrm{Map}_{\mathrm{An}_*}(B, C)) \\ &\simeq \mathrm{Map}_{\mathrm{An}_*}(S^1_+, \mathrm{Map}_{\mathrm{An}_*}(B, C)) \\ &\simeq \mathrm{Map}_{\mathrm{An}_*}(S^1_+ \wedge B, C) \end{aligned}$$

Since B is connected, $S^1_+ \wedge B$ is connected. Thus, we can choose a different base-point.

$$\begin{aligned} LA &\simeq \mathrm{Map}_{\mathrm{An}_*}(S^1_+ \wedge B, C) \\ &\simeq \mathrm{Map}_{\mathrm{An}_*}((S^0 \vee S^1) \wedge B, C) \\ &\simeq A \times \Omega A \end{aligned}$$

□

The following lemma says that the cardinality is compatible with pullback.

Lemma A.7 ([CSY22, Corollary 3.1.14])**.** *Let $\mathcal{C}$ be an ∞-category and $q_1\colon A_1 \to B$ and $q_2\colon A_2 \to B$ be two $\mathcal{C}$-ambidextrous maps. Then for all $X \in \mathcal{C}$, we have*

$$|q_1 \times_B q_2|_X \simeq |q_2|_X \circ |q_1|_X.$$

Assembling the above discussion together, we get the following.

Corollary A.8. *Let $(\mathcal{C}, \otimes, \mathbb{1})$ be an m-semiadditively symmetric monoidal ∞-category. Every m-finite space A is dualizable in $\mathcal{C}$ and we have*

$$\dim_{\mathcal{C}}(A) = |LA|.$$

In particular, if A is a loop space (e.g. $A = B^m C_p$), then

$$\dim_{\mathcal{C}}(A) = |A||\Omega A|.$$

Corollary A.9. *For all $m \in \mathbb{Z}_{\geqslant 0}$, we have*

$$|B^m C_p|_{\widehat{\mathrm{Mod}}_{E_n}} = p^{\binom{n-1}{m}}.$$

Proof. By Corollary A.3 and Corollary A.8 (Note that we already know that $\widehat{\mathrm{Mod}}_{E_n}$ is m-semiadditive when applying this corollary), we have

$$p^{\binom{n}{m}} = \dim_{\widehat{\mathrm{Mod}}_{E_n}}(B^m C_p) = |B^m C_p|_{\widehat{\mathrm{Mod}}_{E_n}} |B^{m-1} C_p|_{\widehat{\mathrm{Mod}}_{E_n}}$$

for all $m \in \mathbb{Z}_{\geqslant 0}$. It is easy to show that $|B^0 C_p| = |C_p| = p$. The formula follows from the formula $\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}$ and the fact that $\pi_0(E_n)$ is torsion-free. □

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