Redshift conjecture for commutative ring spectra

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Abstract

Redshift conjecture concerns about how algebraic K-theory interacts with chromatic homotopy theory. It says that the algebraic K-theory raises the chromatic complexity by 1. The conjecture for commutative ring spectra has been formulated and solved by a series of works by Burklund, Clausen, Hahn, Land, Matthew, Meier, Naumann, Noel, Schlank, Tamme and Yuan. In this note, we introduce and summarize the proof of the redshift conjecture for commutative ring spectra.

Contents

1 The chromatic height and the redshift conjecture

Throughout, we will fix a prime number p implicitly.

It is well-known in chromatic homotopy theory that for a p-local finite spectrum V, if $V \otimes K(n) = 0$, then $V \otimes K(n-1) = 0$, i.e., the chromatic support of V is a ray to the infinity.

In contrast, the chromatic support of $\mathbb{E}_\infty$-ring spectra is an interval or $\mathbb{Z}_{\geqslant} \cup \{\infty\}$. This is due to the following theorem of Hahn.

Theorem 1.1 ([Hah22])**.** *Suppose $R \in \mathrm{CAlg}(\mathrm{Sp})$. If $R \otimes K(n) = 0$, then $R \otimes K(n+1) = 0$.*

The theorem inspires the following definition.

Definition 1.2. For $R \in \mathrm{CAlg}(\mathrm{Sp})$, define ***the chromatic height of*** R to be

$$\mathrm{height}(R) := \inf\{n \geqslant -1 \colon K(n+1) \otimes R = 0\}. \qquad \lrcorner$$

Suppose R is an ordinary commutative ring. Since R is bounded above, $\mathrm{height}(R) = 0$ (cf. [LMMT24, Lemma 2.2(ii)]).

Theorem 1.3 ([Mit90])**.** *For all $n \geqslant 2$, $L_{K(n)} K(R) = 0$.*

Thus, $\mathrm{height}(K(R))$ contains no chromatic information of height $\geqslant 2$. Going from ordinary rings to ring spectra, together with other results, Ausoni-Rognes conjectured that similar phenomenon should happen for higher chromatic heights, i.e., algebraic K-theory increases the chromatic complexity by 1 ([AR02], [AR08]). This is known as the redshift conjecture.

Restricted to commutative ring spectra, we have a well-behaved notion of the chromatic complexity, which is the chromatic height defined above. In this case, the redshift conjecture can be formulated as the following.

Conjecture (Redshift)**.** *For $0 \neq R \in \mathrm{CAlg}(\mathrm{Sp})$,*

$$\mathrm{height}(K(R)) = \mathrm{height}(R) + 1.$$

The proof of the redshift conjecture is a joint work of [BSY22], [CMNN22], [LMMT24] and [Yua21]. We will summarize the proof of this conjecture in this note. To be precise, we prove that $\mathrm{height}(K(R)) \leqslant \mathrm{height}(R) + 1$ in Section 3 and $\mathrm{height}(K(R)) \geqslant \mathrm{height}(R) + 1$ in Section 4 and Section 6.

2 Chromatic backgrounds

Definition 2.1. For any $n \geqslant 0$, a ***type*** n***-complex*** $V(n)$ is a pointed finite CW-complex such that $K(i) \otimes \Sigma^\infty V(n) = 0$ for all $i < n$ and $K(n) \otimes \Sigma^\infty V(n) \neq 0$. By [HS98], if $n > 0$, then there is a self map $v_n \colon \Sigma^d V(n) \to V(n)$ for some $d > 0$, which induces an isomorphism on $K(n)$-homology and nilpotent maps on $K(i)$-homology for $i \neq n$. We call this map a v_n***-self map***.

Let $T(n) := \Sigma^\infty V(n)[v_n^{-1}]$. It is well-known by the thick subcategory theorem that the Bousfield class of $T(n)$ does not depend on the choice of $(V(n), v_n)$ (cf. [LMMT24, Lemma 2.2(vii)]). ⌟

Theorem 2.2. *Suppose $R \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$. Then R is $K(n)$-acyclic if and only if R is $T(n)$-acyclic. In particular, the definition of chromatic height does not matter if we replace $T(n)$ by $K(n)$.*

Proof. If R is $T(n)$-acyclic, then $K(n) \otimes T(n) \otimes R = 0$. Since $K(n)$ is a field spectrum and $K(n) \otimes T(n) \neq 0$, $K(n) \otimes R = 0$.

For the converse, we may assume that $T(n)$ is an $\mathbb{E}_1$-ring spectrum by replacing $V(n)$ by $W(n) := V(n) \otimes DV(n) \simeq \mathrm{End}(\Sigma^\infty V(n))$. The self-map v_n defines a self-map w_n on $W(n)$. By [HS98, Theorem 11], a power of w_n is in the center of $\pi_*(W(n))$, so $W_n[w_n^{-1}]$ can be promoted to an $\mathbb{E}_1$-ring spectrum.

If R is $K(n)$-acyclic, then $T(n) \otimes R$ is $K(m)$-acyclic for all $m \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ since $T(n)$ is $K(m)$ acyclic for $m \in (\mathbb{Z}_{\geqslant 0} - \{n\}) \cup \{\infty\}$. By [HS98, Theorem 3], the unit map of $T(n) \otimes R$ is nilpotent, so $T(n) \otimes R = 0$. □

Definition 2.3. We denote the Bousfield localization with respect to $T(0) \oplus \cdots \oplus T(n)$ by $L_n^{p,f}$. Let $\mathcal{C}_{>n}$ be the thick subcategory of $\mathrm{Sp}_{(p)}^\omega$ consisting of finite spectra of type greater than n. ⌟

Lemma 2.4 (cf. [LMMT24, Lemma 2.6])**.** *The category of $L_n^{p,f}$-acyclic spectra is $\mathrm{Ind}(\mathcal{C}_{>n})$. Consequently, $L_n^{p,f}$ is smashing.*

Lemma 2.5. *For integers $0 \leqslant m < n$ and a spectrum X, there is a pullback diagram*

$$\begin{array}{ccc} L_n^{p,f} X & \longrightarrow & L_{T(m+1)\oplus\cdots\oplus T(n)} X \\ \downarrow & & \downarrow \\ L_m^{p,f} X & \longrightarrow & L_m^{p,f} L_{T(m+1)\oplus\cdots\oplus T(n)} X \end{array}$$

Since $L_m^{p,f}$ is smashing, the lemma is a direct corollary of the following lemma.

Lemma 2.6. *Suppose $E, F \in \mathrm{Sp}$ and L_E preserves F-acyclic spectra, there is a pullback diagram*

$$\begin{array}{ccc} L_{E\oplus F}X & \longrightarrow & L_F X \\ \downarrow & & \downarrow \\ L_E X & \longrightarrow & L_E L_F X \end{array}$$

Proof. Suppose $\tilde{X}$ is the pullback of this diagram without the left-top corner. It suffices to show that $\tilde{X}$ is $(E\oplus F)$-local and has the same $(E\oplus F)$-homology with X.

Pick any $Y \in \mathrm{Sp}$ that is $(E\oplus F)$-acyclic, Y is both E-acyclic and F-acyclic, so $\mathrm{Map}(Y, L_E X) = \mathrm{Map}(Y, L_F X) = \mathrm{Map}(Y, L_E L_F X) = 0$. By the universal property of pullback, every map $Y \to \tilde{X}$ corresponds to a commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & L_F X \\ \downarrow & & \downarrow \\ L_E X & \longrightarrow & L_E L_F X \end{array}$$

in which the maps $Y \to L_E X$ and $Y \to L_F X$ are nullhomotopic and the homotopy between the two paths is also trivial due to above vanishing result. Thus, $\mathrm{Map}(Y, \tilde{X}) = 0$.

Now we prove that $\tilde{X}\otimes(E\oplus F) \simeq X\otimes(E\oplus F)$. From the pullback square, it is clear that $\tilde{X}\otimes E \simeq L_E X\otimes E \simeq X\otimes E$. Since L_E preserves F-acyclic spectra, $L_E X\otimes F \simeq L_E L_F X\otimes F$ via the canonical map $X \to L_F X$. Therefore, $\tilde{X}\otimes F \simeq L_F X\otimes F \simeq X\otimes F$. $\square$

As an algebraic analog of $L_n^{p,f} := L_{T(0)\oplus\cdots\oplus T(n)}$, we have another localization functor $L_n := L_{K(0)\oplus\cdots\oplus K(n)}$. By [Rav84, 2.1(d)], $K(0)\oplus\cdots\oplus K(n)$ is Bousfield equivalent to a ring spectrum called the Morava E-theory.

Throughout, let k be a perfect field of characteristic p and $\mathbb{G}_0$ be a formal group of height $n \geqslant 1$ over k. Let $\mathrm{Perf}_k \subset \mathrm{CRing}$ be the category of (ordinary) perfect algebras over k and $(-)^\sharp, (-)^\flat\colon \mathrm{CRing}_k \to \mathrm{Perf}_k$ be the left and right adjoint of the inclusion functor respectively, i.e., the colimit and limit perfection. For any even periodic complex orientable ring spectrum R, let $\mathfrak{m}$ be the nth Landweber ideal in $\pi_0 R$, generated by any choice of $p, v_1, v_2, \cdots, v_{n-1}$.

Theorem 2.7 ([Lur18, Theorem 5.0.2, 5.1.5])**.** *There is a fully faithful functor*

$$E(-)\colon \mathrm{Perf}_k \to \mathrm{CAlg}(\mathrm{Sp}_{K(n)})$$

such that for any even periodic complex orientable $R \in \mathrm{CAlg}(\mathrm{Sp}_{K(n)})$, $\mathrm{Map}(E(A), R)$ is canonically isomorphic to the set of pairs (i, α), where $i\colon A \to \pi_0(R)/\mathfrak{m}$ is a ring homomorphism and $\alpha\colon i^*(\mathbb{G}_0)_A \cong (\mathbb{G}_R)_{\pi_0(R)/\mathfrak{m}}$ is an isomorphism between formal groups over $\pi_0(R)/\mathfrak{m}$, where $\mathbb{G}_R$ is the formal group associated to R.

Moreover, $\pi_*(E(A)) \cong W(A)[\![u_0, \cdots, u_{n-1}]\!][u^{\pm 1}]$, where $|u| = -2$ and v_i can be chosen as $u_i u^{-p^i+1}$.

Remark 2.8. By [Lur10, Lecture 23, Proposition 2], the Bousfield class of $E(A)$ is the same with $K(0) \oplus \cdots \oplus K(n)$, which is independent on A. ⌟

For any $R \in \mathrm{CAlg}^{\wedge}_{E(k)} := \mathrm{CAlg}_{E(k)}(\mathrm{Sp}_{T(n)})$, let $R^\flat := (\pi_0 R/\mathfrak{m})^\flat$.

Theorem 2.9 ([BSY22, Theorem 2.38])**.** *There is an adjunction*

$$E(-)\colon \mathrm{Perf}_k \rightleftarrows \mathrm{CAlg}^{\wedge}_{E(k)}\colon (-)^\flat.$$

Remark 2.10. Let us give a sketch proof of Theorem 1.1 here.

Firstly, let E_{n+1} be some Morava E-theory E_{n+1} of height $n+1$. Since $K(n+1) \otimes E_{n+1} \neq 0$ and $K(n+1)$ is a field spectrum, R is $K(n+1)$-acyclic if and only if $L_{K(n+1)}(R \otimes E_{n+1}) = 0$. Since $K(n) \otimes L_{K(n+1)}(R \otimes E_{n+1})$ is a module over $K(n) \otimes R$, $L_{K(n+1)}(R \otimes E_{n+1})$ is $K(n)$-acyclic if $R \otimes K(n) = 0$. Therefore, we may assume that R is a $K(n+1)$-local E_{n+1}-algebra.

By the Theorem 2.2, $R \otimes K(n) = 0$ if and only if $R \otimes T(n) = 0$, so u_n is nilpotent in $R/(p, u_1, \cdots, u_{n-1})$, say $u_n^k = 0$. Note that u_n comes from $\pi_0(E_{n+1})$. Hahn proved that the ring $\pi_0(E_{n+1})/(p, u_1, \cdots, u_{n-1})$ is a DVR and the weight p total power operation associated to the group C_p reduces the valuation. Making use of the power operation, Hahn constructed an element in $\pi_0(E_{n+1})/(p, u_1, \cdots, u_{n-1})$ that has smaller valuation than u_n^k and is also zero in $R/(p, u_1, \cdots, u_{n-1})$. Iterating this process, we can show that there is a unit in $\pi_0(E_{n+1}/(p, u_1, \cdots, u_{n-1}))$ mapping to 0 in $R/(p, u_1, \cdots, u_{n-1})$, so $R/(p, u_1, \cdots, u_{n-1}) = 0$. Since R is $K(n+1)$-local, $R = 0$. ⌟

3 Purity in chromatic algebraic K-theory

In this section, we prove that algebraic K-theory raises the chromatic height no more than 1. Firstly, we have the following witness of this principle.

Theorem 3.1 ([CMNN22, Theorem 4.10])**.** *For $n \geqslant 2$,*

$$L_{T(n)}K(L_{n-2}^{p,f}\mathbb{S}) = 0.$$

The above theorem shows that $L_{T(n)}K$ vanishes on elements in $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}_{T(0)\oplus\cdots T(n-2)})$ since every object in this category can be promoted to an algebra over $L_{n-2}^{p,f}\mathbb{S}$. The purity theorem for chromatic algebraic K-theory says that $L_{T(n)}K$ only cares about chromatic heights $(n-1)$ and n.

Theorem 3.2 (Purity theorem)**.** *Suppose $R \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$. For $n \geqslant 1$, the canonical map $R \to L_{T(n-1)\oplus T(n)}R$ induces an equivalence on $L_{T(n)}K(-)$.*

Remark 3.3. This does not imply that $L_{T(n)}K(L_{T(n+1)}\mathbb{S}) = 0$. For example, by Theorem 4.1, $K(E_{n+1})$ has height $(n+2)$ for any Morava E-theory of height $(n+1)$, so $L_{T(n)}K(E_{n+1}) \neq 0$ by Theorem 1.1. ⌟

Thanks to this profound result, we can easily show that algebraic K-theory raises the chromatic height no more than 1.

Proposition 3.4. *For $R \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$,*

$$\mathrm{height}(K(R)) \leqslant \mathrm{height}(R) + 1.$$

Proof. Suppose $\mathrm{height}(R) = n$, so $L_{T(n+1)\oplus T(n+2)}R = 0$. By Theorem 3.2,

$$L_{T(n+2)}K(R) \simeq L_{T(n+2)}K(L_{T(n+1)\oplus T(n+2)}R) = 0.$$ □

Now we want to give a sketch proof for the purity theorem based on Theorem 3.1. Recall from Lemma 2.5 that we can deduce information about $L_{T(n-1)\oplus T(n)}$ from $L_{n-2}^{p,f}$ and $L_n^{p,f}$.

Theorem 3.5 ([LMMT24, Theorem A(1)])**.** *Suppose $R \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$. For $n \geqslant 1$, the canonical map $R \to L_n^{p,f}R$ induces an equivalence on $T(n)$-local K-theory.*

Since algebraic K-theory is a localizing invariant, the theorem is equivalent to say that the kernel of $\mathrm{Perf}(R) \to \mathrm{Perf}(L_n^{p,f}R)$ vanishes after applying $L_{T(n)}K$. This is done by the following two lemmas.

Lemma 3.6. *For any $R \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$, there is a localizing sequence*

$$\mathcal{C}_{>n} \otimes \mathrm{Perf}(R) \to \mathrm{Perf}(R) \to \mathrm{Perf}(L_n^{p,f} R).$$

Moreover, for any $X \in \mathcal{C}_{>n} \otimes \mathrm{Perf}(R)$, $L_n^{p,f} \mathrm{map}(X, X) = 0$.

Proof. By [Nee92, Theorem 2.1] and Lemma 2.4, we have the following fiber sequence in $\mathrm{Cat}_\infty^{\mathrm{perf}}$

$$\mathcal{C}_{>n} \to \mathrm{Perf}(\mathbb{S}) \to \mathrm{Perf}(L_n^{p,f}\, \mathbb{S}).$$

Tensoring the above sequence with $\mathrm{Perf}(R)$, we get the required sequence using the fact that $L_n^{p,f}$ is smashing.

For any $X \in \mathcal{C}_{>n} \otimes \mathrm{Perf}(R)$, X is generated by elements of the form $V \otimes R$ for some $V \in \mathcal{C}_{>n}$. Note that $\mathrm{map}(V \otimes R, V \otimes R) \simeq DV \otimes V \otimes R$ is still $L_n^{p,f}$-acyclic. Thus, $\mathrm{map}(X, X)$ is also $L_n^{p,f}$-acyclic. □

Lemma 3.7 ([LMMT24, Proposition 3.6])**.** *Suppose $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$ and $L_n^{p,f} \mathrm{Map}(X, X) = 0$ for all $X \in \mathcal{C}$. Then $L_{T(i)} K(\mathcal{C}) = 0$ for all $1 \leqslant i \leqslant n$.*

Sketch proof. Since v_n-self map is in positive degree for $n > 0$, the v_n-periodic homotopy groups vanish for bounded above spectra. Thus, $T(i)$-equivalence can be verified after truncation for $i > 0$. From this and the plus construction $\Sigma^\infty \mathrm{BGL}(R) \xrightarrow{\simeq} \Sigma^\infty \Omega^\infty \tau_{\geqslant 1} K(R)$, we may deduce that Theorem 3.5 is true for highly connected R ([LMMT24, Proposition 3.1]). Furthermore, by some arguments on localizing invariants, we can show that for any connective $L_n^{p,f}$-acyclic ring spectrum R, $L_{T(i)} K(R) \simeq L_{T(i)} K(\pi_0 R)$ and $L_{T(i)} K(\mathbb{Z}/p^k) = 0$ for $k \geqslant 0$([LMMT24, Proposition 3.4 and Corollary 3.5]).

Since algebraic K-theory commutes with filtered colimits, we may assume that $\mathcal{C}$ is generated under finite direct sums and retracts by one object X. Let $\hom(X, X)$ denotes the connective ring spectrum corresponding to $\mathrm{Map}(X, X)$, which can be promoted to a group-like $\mathbb{E}_\infty$-anima since $\mathcal{C}$ is additive. By the additive version of the Schwede-Shipley theorem, $\mathcal{C} \simeq \mathrm{Proj}^\omega(\hom(X, X))$. Let K^{add} be the group-completion K-theory. Then $K^{\mathrm{add}}(\mathcal{C}) \simeq \tau_{\geqslant 0} K(\hom(X, X))$ is $T(i)$-acyclic by the above paragraph and the assumption that $\hom(X, X)$ is $L_n^{p,f}$-acyclic (in particular, it is p-power torsion).

Suppose $S_\bullet \mathcal{C} \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Cat}_\infty^{\mathrm{perf}})$ is the Waldhausen $S_\bullet$-construction. There is an equivalence of connective spectra

$$(\tau_{\geqslant 0}K(\mathcal{C}))[1] \simeq |K^{\mathrm{add}}(S_\bullet \mathcal{C})|.$$

By the above discussion and the explicit construction of the Waldhausen $S_\bullet$-construction, the right-hand side is $T(i)$-acyclic, so is $K(\mathcal{C})$. □

Now we can finish the proof of the purity theorem.

Proof of Theorem 3.2. By Lemma 2.5, we have the following pullback diagram.

$$\begin{array}{ccc} L_n^{p,f}R & \longrightarrow & L_{T(n-1)\oplus T(n)}R \\ \downarrow & & \downarrow \\ L_{n-2}^{p,f}R & \longrightarrow & L_{n-2}^{p,f}L_{T(n-1)\oplus T(n)}R \end{array}$$

By [LT19, Theorem A] and the fact that $L_{n-2}^{p,f}$ is smashing, the diagram is still a pullback diagram after applying $L_{T(n)}K$. Note that the bottom row vanishes by Theorem 3.1 after applying $L_{T(n)}K$. Therefore,

$$L_{T(n)}K(R) \xrightarrow{\simeq} L_{T(n)}K(L_n^{p,f}R) \xrightarrow{\simeq} L_{T(n)}K(L_{T(n-1)\oplus T(n)}R),$$

where the first map is induced by the canonical map and is an equivalence by Theorem 3.5. □

4 Redshift for Morava E-theories

In his paper [Yua21], Yuan proved the following interesting result, producing an example of redshift for Morava E-theories.

Theorem 4.1. *We have that* $L_{T(n+1)}K(E(k)) \neq 0$.

Recall firstly that we have the following blueshift result says that $(-)^{tC_p}$ reduces the chromatic height by 1.

Theorem 4.2 (Chromatic blueshift, [Kuh04])**.** *Let* $X \in \mathrm{Sp}_{T(n)}^{BC_p}$. *Then* $L_{T(n)}X^{tC_p} = 0$.

The idea of proving Theorem 4.1 is that although $(-)^{tC_p}$ lower the chromatic height by 1, taking $(-)^{hS^1/C_p}$ with respect to the residual S^1/C_p-action on $(-)^{tC_p}$ gives back the chromatic height by the Tate orbit lemma. The latter object is related to the algebraic K-theory via the Dennis trace map to THH. The following example illustrates part of the idea.

Example 4.3. Suppose $X = \mathbb{Z}$. Then $\pi_*\mathbb{Z}^{tC_p} \cong \mathbb{F}_p((T))$, where $|T| = -2$. Since $v_0 = p$ vanishes in $\mathbb{Z}^{tC_p}$, $\mathrm{height}(\mathbb{Z}^{tC_p}) = -1$.

However, if we consider the residual C_{p^2}/C_p-action on $\mathbb{Z}^{tC_p}$ and take the fixed point, we get

$$\pi_*(\mathbb{Z}^{tC_p})^{hC_{p^2}/C_p} \cong \mathbb{Z}/p^2((T)), \quad |T| = -2,$$

by comparing the homotopy fixed point spectral sequence associated to it with the one associated to $(\mathbb{Z}^{hC_p})^{hC_{p^2}/C_p} \simeq \mathbb{Z}^{hC_{p^2}}$. Similarly, if we add more p-division values into it, we will get more powers of p:

$$\pi_*(\mathbb{Z}^{tC_p})^{hC_{p^k}/C_p} \cong \mathbb{Z}/p^k((T)), \quad |T| = -2,$$
$$\pi_*(\mathbb{Z}^{tC_p})^{hS^1/C_p} \cong \mathbb{Z}_p((T)), \quad |T| = -2. \quad \lrcorner$$

This phenomenon is a special case of the following Tate orbit lemma.

Lemma 4.4 (Tate orbit lemma, [NS18, Lemma II.4.2])**.** *Suppose X is a bounded below spectrum with S^1-action. The map $X^{tS^1} \to (X^{tC_p})^{hS^1/C_p}$ exhibits the target as the p-completion of the source.*

Therefore, it is natural to prove the following result.

Proposition 4.5. *Let R be a homotopy commutative ring spectrum and $n \geqslant 1$ such that $L_{T(n)}R \neq 0$. Then $L_{T(n)}R^{tS^1} \neq 0$.*

Proof. If R is complex oriented, then

$$\pi_*R^{hS^1} \cong R^*(BS^1) \cong R_*[\![T]\!], \quad |T| = -2$$

and $R^{tS^1} \simeq R^{hS^1}[T^{-1}]$, which contains R itself as a direct summand as an R-module. Hence, $L_{T(n)}R^{tS^1} \neq 0$.

Note that $K(n)$ is complex oriented, so is $R \otimes K(n)$. By Theorem 2.2, we have that $L_{T(n)}R \neq 0$ if and only if $L_{K(n)}R \neq 0$, if and only if $L_{K(n)}(R \otimes K(n)) \neq 0$, if and only if $L_{T(n)}(R \otimes K(n)) \neq 0$. By the above discussion, $L_{T(n)}(R \otimes K(n))^{tS^1} \neq 0$. Since there is a ring map $R \to R \otimes K(n)$, $L_{T(n)}R^{tS^1} \neq 0$. $\square$

It follows from the Tate orbit lemma that $L_{T(n)}(R^{tC_p})^{hS^1/C_p} \neq 0$ for bounded below R. Now we can prove that the redshift phenomenon happens for R^{tC_p}.

Theorem 4.6. *Suppose $R \in \mathrm{CAlg}(\mathrm{Sp})$, $n \geqslant 1$ and $L_{T(n)}R \neq 0$. Then $L_{T(n)}K(R^{tC_p}) \neq 0$.*

Proof. Since the Tate orbit lemma only works for bounded below spectra, we need to pass to connective case at first. Let $r \to R$ be the connective cover of R. We want to show that $L_{T(i)}r^{tC_p} \simeq L_{T(i)}R^{tC_p}$ for all $i \geqslant 0$. It suffices to show that $L_{T(i)}(\tau_{\leqslant -1}R)^{tC_p} = 0$. Note that $L_{T(i)}$ and $(-)^{tC_p}$ commutes with the colimit of the Whitehead tower (cf. [NS18, Lemma I.2.6]). Thus, we are reduced to show that $L_{T(i)}M^{tC_p}$ for any $M \in \mathrm{Ab}$. Indeed, $M^{tC_p} \in \mathrm{Mod}_{\mathbb{Z}}$ is p-torsion, so it is $T(i)$-acyclic.

Consider the residual $S^1 \simeq S^1/C_p$-action on the target of the identity map $r^{tC_p} \to r^{tC_p}$. This map extends uniquely to an S^1-equivariant map $THH(r^{tC_p}) \to r^{tC_p}$. Taking the S^1-fixed points and composing it with the S^1-invariant Dennis trace map, we get a map of $\mathbb{E}_\infty$-rings

$$K(r^{tC_p}) \to \mathrm{THH}(r^{tC_p})^{hS^1} \to (r^{tC_p})^{hS^1/C_p}.$$

The codomain is not $T(n)$-acyclic by the previous proposition, so $L_{T(n)}K(r^{tC_p}) \neq 0$. Finally, by the purity theorem and the fact that $L_{T(n-1)\oplus T(n)}r^{tC_p} \simeq L_{T(n-1)\oplus T(n)}R^{tC_p}$, we get that $L_{T(n)}K(R^{tC_p}) \neq 0$. □

Remark 4.7. The theorem also shows that the purity theorem is kind of optimal, i.e., $L_{T(n)}$-localization does not induce an equivalence on $L_{T(n)}K$. Suppose $R \in \mathrm{CAlg}(\mathrm{Sp})$ is of height n. Then $L_{T(n)}R^{tC_p} = 0$ by Kuhn's blueshift theorem, while $L_{T(n)}K(R^{tC_p}) \neq 0$ by the above theorem.

In fact, we can show that $L_{T(n-1)}$-localization also does not induce an equivalence on $L_{T(n)}K$ as well, so the purity theorem is really optimal ([LMMT24, Remark 3.11]). ⌟

Naively, in order to prove Theorem 4.1, we want to construct an $\mathbb{E}_\infty$-map $E(k) \to E_{n+1}^{tC_p}$ for some Morava E-theory of height $(n+1)$, which corresponds to a deformation of $\mathbb{G}_0$ to $\pi_0(E_{n+1}^{tC_p})$ by Theorem 2.7. However, $\pi_0(E_{n+1}^{tC_p}) \cong \pi_0(E_{n+1})(\!(T)\!)/[p](T)$ is a quotient ring of Laurent series and the corresponding formal group law is complicated ([AMS98]).

A remedy for this situation is that although $\pi_0(E_{n+1}^{tC_p})$ itself is complicated, we can zoom into a localization of it that is quite easy to handle and is also non-vanishing after applying $L_{T(n)}K$. In particular, if the residue field of the localization is separably closed, all formal groups laws of the same height are isomorphic by Lazard's theorem, so it is easy to construct a deformation. This is done by the following lemma.

Lemma 4.8 ([Yua21, Lemma 4.5]). *Suppose $R \in \mathrm{CAlg}(\mathrm{Sp})$ and $L_{T(n)}K(R) \neq 0$. Then there exists a prime ideal $\mathfrak{p} \subset \pi_0(R)$ such that $L_{T(n)}K(R^{\mathrm{sh}}_{\mathfrak{p}}) \neq 0$, where $R^{\mathrm{sh}}_{\mathfrak{p}}$ is the strict henselization of R at $\mathfrak{p}$. In particular, $\pi_0(R^{\mathrm{sh}}_{\mathfrak{p}})$ is a local ring with separably closed residue field.*

For the detailed proof of Theorem 4.1, see [Yua21, Theorem A].

5 Coverings by Morava E-theories

Inspired by the idea of the proof of Theorem 4.1, in order to prove the redshift conjecture for arbitrary $R \in \mathrm{CAlg}(\mathrm{Sp})$, we want to create a ring map $R \to E_n$ for some Morava E-theory E_n. This is done by the following main result of [BSY22].

Theorem 5.1 ([BSY22, Theorem 5.1]). *Suppose $R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$. Then there is an $A \in \mathrm{Perf}_k$ of Krull dimension 0 and a nilpotence detecting map $R \to E(A)$ in $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$.*

To prove the theorem, we first introduce what does 'nilpotence detecting' mean.

Definition 5.2. A ***locally rigid category*** is a compactly generated symmetric monoidal stable category such that every compact object is dualizable. Let $\mathrm{Pr}^{\mathrm{rig}} \subset \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{st}})$ be the (non-full) subcategory of locally rigid categories with compact preserving functors. ⌟

Definition 5.3. Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{rig}}$ and $f\colon R \to X \in \mathrm{CAlg}(\mathcal{C})$. We say that f ***detects nilpotence*** if for any $C \in \mathrm{Mod}_R(\mathcal{C})^{\omega}$ and $g\colon C \to D \in \mathrm{Mod}_R(\mathcal{C})$, g is smash nilpotent if and only if g is smash nilpotent after base-changing to X, i.e., $g \otimes X$ is smash nilpotent in $\mathrm{Mod}_X(\mathcal{C})$. ⌟

The proof relies on a technique called the 'small object argument'.

Definition 5.4. Let $\mathcal{C}$ be an ∞-category and S be a collection of morphisms in $\mathcal{C}$. We say that S is ***weakly saturated*** if

1. S is closed under cobase-change. That is, for any pushout square

$$\begin{array}{ccc} C & \longrightarrow & R \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f'} \\ D & \longrightarrow & X \end{array}$$

in $\mathcal{C}$ such that $f \in S$, we have $f' \in S$.

2. S is closed under retracts in $\mathcal{C}^{\Delta^1}$.

3. S is closed under transfinite composition. That is, for any ordinal α and functor $F\colon \alpha \to \mathcal{C}$ such that

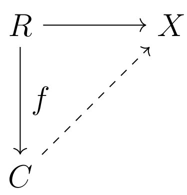
   (a) for any non-zero limit ordinal $\beta < \alpha$, the diagram $F|_{\beta+1}$ is a colimit diagram, and

   (b) for any ordinal β such that $\beta + 1 < \alpha$, the map $F(\beta) \to F(\beta+1)$ is in S,

   we have $F(0) \to F(\beta)$ is in S for all $\beta < \alpha$.

Theorem 5.5 ([BSY22, Theorem 4.36])**.** *Let $\mathcal{C} \in \mathrm{Pr}^{\mathrm{rig}}$. The collection of nilpotence detecting maps is weakly saturated in $\mathrm{CAlg}(\mathcal{C})$.*

Definition 5.6. Let $\mathcal{C}$ be an ∞-category. Let $X \in \mathcal{C}$ and $f\colon R \to C$ be a morphism in $\mathcal{C}$. We say that f has the ***right lifting property with respect to*** X, denoted as $f \perp X$, if for any map $R \to X$, we have a lift $C \to X$ as in the following diagram.



Proposition 5.7 (The small object argument, [Lur11, Proposition 1.4.7], cf. [BSY22, Proposition 4.35])**.** *Let $\mathcal{C}$ be a presentable ∞-category, S be a weakly saturated class of morphisms in $\mathcal{C}$ and $S_0 \subset S$ be a set of morphisms in S. Then for any $R \in \mathcal{C}$, there is a morphism $R \to X_R$ in $\mathcal{C}$ such that*

1. *$R \to X_R$ is in S, and*

2. *for any $f \in S_0$, $f \perp X_R$.*

In order to prove Theorem 5.1, we want to apply the small object argument to produce the required nilpotence detecting map $R \to X_R$. We choose that subset S_0 properly such that the right lifting property ensures that X_R in Proposition 5.7(2) ensures that X_R is equivalent to some Morava E-theory of a perfect algebra of Krull dimension 0.

Firstly, the map $\mathbb{1}_{T(n)} \to E(k)$ detects nilpotence for any height n Morava E-theory by [HS98]. Since nilpotence detecting maps are closed under composition by Theorem 5.5, we may assume that $R \in \mathrm{CAlg}^{\wedge}_{E(k)}$.

Proposition 5.8 ([BSY22, Proposition 5.9 and 5.11]). *Let $E(k)\{z^i\}$ be the free commutative algebra with a class in degree i.*

There are the following three nilpotence detecting maps with the corresponding properties in $\mathrm{CAlg}^{\wedge}_{E(k)}$:

1. *$f\colon E(k[T^{1/p^\infty}]) \to E(k[T^{\pm 1/p^\infty}]) \times E(k)$ such that $f \perp X_R$ if and only if $X_R^\flat$ is of Krull dimension 0, if and only if the counit map $E(X_R^\flat) \to X_R$ in Theorem 2.9 is injective on π_0,*
2. *$g\colon E(k)\{z^0\} \to E(B)$ with $B := (\pi_0(E(k)\{z^0\})/\mathfrak{m})^\sharp$ such that $g \perp X_R$ if and only if the counit map $E(X_R^\flat) \to X_R$ is surjective on π_0,*
3. *$h\colon E(k)\{z^1\} \xrightarrow{z^1 \mapsto 0} E(k)$ such that $h \perp X_R$ if and only if $\pi_1 X_R = 0$.*

Consequently, let $S_0 := \{f, g, h\}$. The element X_R produced by the small object argument is isomorphic to $E(X_R^\flat)$ and $X_R^\flat \in \mathrm{Perf}_k$ is of Krull dimension 0. Therefore, we have proved Theorem 5.1.

Remark 5.9. According to [Lur11, Proposition 1.4.7], the map $R \to X_R$ is a transfinite pushout of maps in S_0. Intuitively, the map $A \to X_A$ is given by iteratively taking pushouts with maps in S_0 so that it factors through all maps in S_0, which is the condition 2 in Proposition 5.7. ⌟

Remark 5.10. Heuristically, we can explain the maps f, g, h as follows.

1. A diagram

$$\begin{array}{ccc} E(k[T^{1/p^\infty}]) & \to & X_R \\ \downarrow f & \nearrow & \\ E(k[T^{\pm 1/p^\infty}]) \times E(k) & & \end{array}$$

may be seen as a diagram

$$\begin{array}{ccc} k[T^{1/p^\infty}] & \longrightarrow & X_R^\flat \\ (T \mapsto T, T \mapsto 0) \downarrow & \nearrow & \\ k[T^{\pm 1/p^\infty}] \times k & & \end{array}$$

inspired by Theorem 2.9. Geometrically, any map $k[T^{1/p^\infty}] \to X_R^\flat$ factors through $k[T^{\pm 1/p^\infty}] \times k$ implies that $X_R^\flat$ is so discrete that it can only be Krull dimension 0. For the second claim, [BSY22, Corollary 3.51] implies that it suffices to show that $X_R^\flat \to (\pi_0 X_R)/\mathfrak{m}$ is injective. Pick any element $x \in X_R^\flat$ maps to zero in $(\pi_0 X_R)/\mathfrak{m}$, each component of x is nilpotent. Since $X_R^\flat$ is a perfect algebra of Krull dimension 0, x is a unit multiple of an idempotent e. Thus, every component of e is also nilpotent. Since e is idempotent, $e = 0$, so $x = 0$.

2. Due to the definition of $E(k)\{z^0\}$, a map $E(k)\{z^0\} \to X_R$ corresponds to an element x in $\pi_0(X_R)$. By Theorem 2.9, a map $E(B) \to X_R$ corresponds to a map $(\pi_0(E(k)\{z^0\})/\mathfrak{m})^\sharp \to X_R^\flat$, which corresponds to a map $E(k)\{z^0\} \to E(X_R^\flat)$ by [BSY22, Theorem 3.4]. Therefore, factoring through g implies that we can lift x to $\pi_0 E(X_R^\flat)$.

3. Since $E(k)\{z^1\}$ is the free commutative algebra over $E(k)$ with a class in degree 1, factoring through h means that there is no obstruction in degree 1, i.e., $\pi_1 X_R = 0$. ⌟

6 Redshift

The following is an easy corollary of Theorem 5.1.

Corollary 6.1. *Suppose $0 \neq R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$. There is an algebraically closed field L and a map $R \to E(L)$ in $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$.*

Having done with this last step, we are finally ready to finish the proof of the redshift conjecture in the full generality.

Theorem 6.2 (Redshift)**.** *For $0 \neq R \in \mathrm{CAlg}(\mathrm{Sp})$,*

$$\mathrm{height}(K(R)) = \mathrm{height}(R) + 1.$$

Proof. Suppose $\mathrm{height}(R) = n$. In light of Proposition 3.4, we only need to show that $L_{T(n+1)}K(R) \neq 0$. By Corollary 6.1 and the fact that R has height n, we have a map $L_{T(n)}R \to E(L)$ in $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$ for some algebraically closed field L. Therefore, we get a composition of maps in $\mathrm{CAlg}(\mathrm{Sp}_{T(n+1)})$,

$$L_{T(n+1)}K(R) \to L_{T(n+1)}K(L_{T(n)}R) \to L_{T(n+1)}K(E(L)).$$

By Theorem 4.1, the target is non-trivial. Thus, the domain is non-trivial since the only algebra over the zero algebra is zero. $\square$

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